

Code loops: automorphisms and representations

Alexandre Grichkov and Rosemary M. Pires

December 9, 2014

Abstract

In this work we construct free Moufang loop in the variety generated by code loops. We apply this construction for study the code loops. Moreover, we define and determine all basic representations of code loops of rank 3 and 4.

1 Introduction

The study of code loops began with the paper [2] of Griess. Let's recall the definition of code loop. First let \mathbf{F}_2^n be a n -dimensional vector space over a field with two elements $\mathbf{F}_2 = \{0, 1\}$. For vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ we define $|v| = |\{i | v_i = 1\}|$ and $|u \cap v| = |\{i | u_i = v_i = 1\}|$.

A double even code is a subspace $V \subseteq \mathbf{F}_2^n$ such that $|v| \equiv 0 \pmod{4}$ and $|u \cap v| \equiv 0 \pmod{2}$ for all $u, v \in V$.

Let V be a double even code and $L(V)$ be the set $\{1, -1\} \times V$. In [2] R.Griess proved that there exists a function $\phi : V \times V \rightarrow \{1, -1\}$, called factor set, such that

$$\begin{aligned}\phi(v, v) &= (-1)^{\frac{|v|}{4}}, \\ \phi(v, w) &= (-1)^{\frac{|v \cap w|}{2}} \phi(w, v), \\ \phi(0, v) &= \phi(v, 0) = 1, \\ \phi(v + w, u) &= \phi(v, w + u) \phi(v, w) \phi(w, u) (-1)^{|v \cap w \cap u|},\end{aligned}\tag{1}$$

where $|v \cap w \cap u|$ denotes the number of positions in which the coordinates of u , v and w are both nonzero.

In order to define code loop, let $v, w \in V$ and $\phi(v, w) \in \{1, -1\}$, and let $(1, v)$ denotes v

and $(-1, v)$ denotes $-v$. We define a product “ \cdot ” em $L(V)$ by:

$$\begin{aligned} v.w &= \phi(v, w)(v + w), \\ v.(-w) &= (-v).w = -(v.w), \\ (-v).(-w) &= v.w. \end{aligned} \tag{2}$$

With the product defined above, Griess (1986, [2]) proved that $(L(V), \cdot)$, or merely $L(V)$, has a Moufang loop structure. The Moufang loop $L(V)$ is called code loop. We say that $L(V)$ has rank m , if $\dim_{\mathbf{F}_2} V = m$.

Moreover, Chein and Goodaire (1990, [4]) proved that code loops have a unique nonidentity square, a unique nonidentity commutator, and a unique nonidentity associator. In other words, for any $u, v, w \in V$:

$$\begin{aligned} v^2 &= (-1)^{\frac{|v|}{4}} 0, \\ [u, v] &= u^{-1}v^{-1}uv = (-1)^{\frac{|u \cap v|}{2}} 0, \\ (u, v, w) &= ((uv)w)((u(vw))^{-1}) = (-1)^{|u \cap v \cap w|} 0. \end{aligned} \tag{3}$$

On the other hand, a Moufang loop L is called ***E-loop*** if there is a central subloop Z with 2 elements such that $L/Z \in \mathcal{A}$, where \mathcal{A} is the variety of groups with identity $x^2 = 1$.

Chein e Goodaire (1990, [4]) proved that finite code loops may be characterized as Moufang loops L for which $|L^2| \leq 2$. So, directly from the proof of this result we can obtain that a finite Moufang loop L is a code loop if and only if L is a *E-loop*.

In this paper, we study the representations of code loops. In Section 2, we prove that code loops of rank n can be characterized as a homomorphic image of a certain free Moufang loops with n generators and we introduce the concept of characteristic vectors associated with code loops. In Section 3, we present the classification of code loops of rank 3 and 4 and their corresponding groups of outer automorphisms. There are exactly 5 nonassociative code loops of rank 3 (up to isomorphism) and 16 nonassociative code loops of rank 4 (up to isomorphism). By using this, in Section 4, we determine all basic representations of code loops of rank 3 and 4.

2 Free Loops of the Variety \mathcal{E}

In the following we introduce the variety of Moufang loops \mathcal{E} , generated by all code loops.

Definition 1. *Let \mathcal{E} be the variety of Moufang loops with the following identities:*

$$\begin{aligned} x^4 = 1, [x, y]^2 = 1, (x, y, z)^2 = 1, \\ [x^2, y] = 1, [[x, y], t] = 1, [(x, y, z), t] = 1, \\ (x^2, y, z) = 1, ([x, y], z, t) = 1, ((x, y, z), t, s) = 1. \end{aligned} \quad (4)$$

We observe that code loops are contained in \mathcal{E} . In fact, a code loop is a Moufang loop with a Unique Nonidentity Commutator (Associator, Square). Besides, Chein and Goodaire ([4], Theorem 1) proved that if L is a Moufang loop with a unique nonidentity square e , then $e^2 = 1$ and either L is an abelian group or else $[L, L] = (L, L, L) = L^2 = \{1, e\} \subseteq \mathcal{Z}(L)$. Here, $[L, L]$, (L, L, L) , and L^2 denotes, respectively, the sets of all commutators $[x, y]$, all associators (x, y, z) and all squares x^2 , $x, y, z \in L$. Besides, the center of L , $\mathcal{Z}(L)$, is the set of elements of L which associate with every pair of elements of L and which commute with every element of L .

Note that those identities are not independents. It is interesting problem to find the minimal set of identities that defines the variety \mathcal{E} (see the Conjecture below).

Conjecture 1. *The variety \mathcal{E} has the following minimal set of identities:*

$$x^4 = 1, [x, y]^2 = 1, [x^2, y] = 1, (x^2, y, z) = 1.$$

Let V be a \mathbf{F}_2 -space with a basis $\{v_1, v_2, \dots\}$. We identify each element of V with a finite subset of the set of natural numbers \mathbb{N} as follows: We consider the biunivocal correspondence $\sigma : V \longrightarrow \mathcal{P}(\mathbb{N})$, where

$$v = a_1 v_1 + a_2 v_2 + \dots \in V \longmapsto \sigma(v) = \sigma = \{i | a_i = 1\} \in \mathcal{P}(\mathbb{N}).$$

In this way, we use the notation $\sigma \in V$ when $\sigma = \sigma(v)$, for $v \in V$. Besides, we observe that, $\sigma(v + w) = \sigma(v) \Delta \sigma(w)$, where $\sigma \Delta \mu = (\sigma \setminus \mu) \cup (\mu \setminus \sigma)$.

We consider $W = V \wedge V$ and $U = V \wedge V \wedge V$ antisymmetric products of V . This means that W has a basis $\{i \wedge j \mid i \wedge j = j \wedge i, i \wedge i = 0; i, j \in \mathbb{N}\}$ and U has a basis $\{i \wedge j \wedge k \mid i \wedge j \wedge k = j \wedge k \wedge i = k \wedge i \wedge j; i, j, k \in \mathbb{N}\}$. Moreover, $i \wedge j \wedge k = 0$ if and only if $|\{i, j, k\}| < 3$. Let \bar{V} be an isomorphic copy of V with a basis $\{\bar{i} \mid i \in \mathbb{N}\}$.

Let the set $\mathcal{F} = V \bar{V} W U$. We want to define a product $\mathcal{F}\mathcal{F} \subseteq \mathcal{F}$ such that for $i, j, k \in \mathbb{N}$ we have

$$i^2 = \bar{i}, [i, j] = i^{-1} j^{-1} i j = i \wedge j, (i, j, k) = (i j k)(i j k)^{-1} = i \wedge j \wedge k. \quad (5)$$

For the following, given an element $\sigma = \{i_1, i_2, \dots, i_s\} \in V$, we identify it in \mathcal{F} with a product $\sigma = (\dots(i_1 i_2) i_3) \dots i_s$. Now we define

$$\sigma.\mu = \sigma\Delta\mu . \prod_{i \in \sigma \cap \mu} \bar{i} . \prod_{i \in \sigma, j \in \mu, i > j} i \wedge j . \prod_{i \in \sigma, j, k \in \mu, j < k} i \wedge j \wedge k, \quad \sigma, \mu \in V. \quad (6)$$

$$x^2 = [x, y] = (x, y, z) = 1, \quad x \in \bar{V}.W.U, \quad y, z \in \mathcal{F}. \quad (7)$$

By definition $\mathcal{Z}(\mathcal{F}) = \bar{V}.W.U$. Now let $v, w \in \mathcal{F}$ be of the form $v = v_0 z_0$ and $w = w_0 z_1$, where $v_0, w_0 \in V$ and $z_0, z_1 \in \mathcal{Z}(\mathcal{F})$. Therefore, the product of v by w is given by: $v.w = (v_0.w_0).z_0.z_1$.

For the following proofs we use the notation more concise, for all $\sigma, \mu, \lambda \in V$:

$$[\sigma, \mu, \lambda] = \prod_{i \in \sigma, j \in \mu, k \in \lambda, j < k} i \wedge j \wedge k \quad (8)$$

$$[\sigma, \mu] = \prod_{i \in \sigma, j \in \mu, i > j} i \wedge j \quad (9)$$

$$\{\sigma, \mu\} = \begin{cases} \prod_{i \in \sigma \cap \mu} \bar{i}, & \text{se } \sigma \cap \mu \neq \emptyset \\ 1, & \text{se } \sigma \cap \mu = \emptyset \end{cases} \quad (10)$$

Lemma 1. *With the above definitions, in \mathcal{F} are valid the following equalities, for all $\sigma, \mu, \lambda \in V$:*

$$\{\sigma, \mu\}\{\sigma, \lambda\}\{\sigma\Delta\mu, \lambda\Delta\sigma\} = \{\lambda, \mu\}\{\sigma, \lambda\Delta\mu\}\{\sigma\Delta\mu\Delta\lambda, \sigma\} \quad (11)$$

$$[\sigma, \mu][\lambda, \sigma][\sigma\Delta\mu, \lambda\Delta\sigma] = [\mu, \lambda][\sigma, \lambda\Delta\mu][\sigma\Delta\mu\Delta\lambda, \sigma] \quad (12)$$

$$[\lambda\Delta\tau, \mu, \sigma] = [\lambda, \mu, \sigma][\tau, \mu, \sigma]$$

$$[\sigma, \lambda\Delta\tau, \mu] = [\sigma, \lambda, \mu][\sigma, \tau, \mu]$$

$$[\sigma, \mu, \lambda\Delta\tau] = [\sigma, \mu, \lambda][\sigma, \mu, \tau] \quad (13)$$

$$[\sigma, \lambda, \sigma][\sigma, \sigma, \lambda] = 1 \quad (14)$$

$$[\sigma, \lambda, \mu][\mu, \lambda, \sigma][\sigma, \mu, \lambda][\mu, \sigma, \lambda] = 1 \quad (15)$$

$$[\sigma\Delta\mu, \lambda\Delta\sigma, \lambda\Delta\sigma][\sigma, \mu, \mu][\lambda, \sigma, \sigma][\sigma\Delta\mu\Delta\lambda, \sigma, \sigma][\sigma, \mu\Delta\lambda, \mu\Delta\lambda][\mu, \lambda, \lambda] = 1 \quad (16)$$

Proof. We denote

$$\xi_1 = \sigma \cap \lambda \cap \mu, \quad \xi_2 = (\sigma \cap \mu) \setminus \lambda, \quad \xi_3 = (\sigma \cap \lambda) \setminus \mu, \quad \xi_4 = \sigma \setminus (\mu \cup \lambda),$$

$$\xi_5 = (\lambda \cap \mu) \setminus \sigma, \quad \xi_6 = \mu \setminus (\sigma \cup \lambda), \quad \xi_7 = \lambda \setminus (\sigma \cup \mu).$$

We write $i, ij = (ij), ijk = (ijk), \{i, j\}, \{ij, k\}, [i, j], [ij, k], [i, j, k]$ and $[ip, j, k]$ instead $\xi_i, \xi_i \cup \xi_j, \xi_i \cup \xi_j \cup \xi_k, \{\xi_i, \xi_j\}, \{\xi_i \cup \xi_j, \xi_k\}, [\xi_i, \xi_j], [\xi_i \cup \xi_j, \xi_k], [\xi_i, \xi_j, \xi_k]$ and $[\xi_i \cup \xi_j, \xi_k]$, respectively.

To prove the equality (11), first we see that

$$\{\sigma \cup \mu, \lambda\} = \{\sigma, \lambda\}\{\mu, \lambda\}, \text{ for all } \sigma, \mu, \lambda \in V. \quad (17)$$

Thus, by the equality (17) and by the fact $\{\sigma, \mu\} = 1$ if $\sigma \cap \mu = \emptyset$, for all $\sigma, \mu \in V$, we can rewrite the right part of (11) as follows:

$$\{1, 1\}\{5, 5\}\{3, 3\}\{2, 2\}\{4, 4\}\{1, 1\} = \{5, 5\}\{3, 3\}\{2, 2\}\{4, 4\}.$$

In fact:

$$1. \{\sigma \Delta \mu \Delta \lambda, \sigma\} = \{\sigma \setminus (\mu \Delta \lambda), \sigma\}\{(\mu \Delta \lambda) \setminus \sigma, \sigma\} = \{\xi_4, \sigma\}\{\xi_1, \sigma\}\{\xi_6, \sigma\}\{\xi_7, \sigma\}.$$

We note that $\sigma \cap \xi_4 = \xi_4$, then $\{\xi_4, \sigma\} = \{\xi_4, \xi_4\} = \{4, 4\}$. Analogously, $\{\xi_1, \sigma\} = \{1, 1\}$.

We also note that $\xi_6 \cap \sigma = \emptyset$ and $\xi_7 \cap \sigma = \emptyset$, so we have $\{\xi_6, \sigma\} = \{\xi_7, \sigma\} = 1$. Therefore, $\{\sigma \Delta \mu \Delta \lambda, \sigma\} = \{4, 4\}\{1, 1\}$.

$$2. \{\sigma, \lambda \Delta \mu\} = \{\sigma \cap (\lambda \setminus \mu), \sigma \cap (\lambda \setminus \mu)\}\{\sigma \cap (\mu \setminus \lambda), \sigma \cap (\mu \setminus \lambda)\} = \{3, 3\}\{2, 2\}.$$

$$3. \text{ Since } \lambda \cap \mu = (\lambda \cap \mu \cap \sigma) \cup ((\lambda \cap \mu) \setminus \sigma), \text{ then } \{\lambda, \mu\} = \{\xi_1, \mu\}\{\xi_5, \mu\} = \{1, 1\}\{5, 5\}.$$

Thus, by (1),(2) and (3), we have

$$\{\lambda, \mu\}\{\sigma, \lambda \Delta \mu\}\{\sigma \Delta \mu \Delta \lambda, \sigma\} = \{5, 5\}\{3, 3\}\{2, 2\}\{4, 4\}.$$

Analogously, we prove that the left part of (11) can be rewritten in the same way.

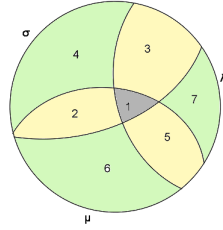


Figure 1: Representative diagram of σ , λ and μ

Now we are going to prove the equality (12). We have $[\sigma \cup \mu, \lambda] = [\sigma, \lambda][\mu, \lambda]$, if $\sigma \cap \mu = \emptyset$. By the Diagram (Fig.1), we can rewrite σ, λ and μ as follows: $\sigma = (1234)$, $\lambda = (1357)$ and $\mu = (1256)$. So the right and left parts of (12) can be rewritten as a product of $[i, j]$, $i, j = 1, \dots, 7$. As $[i, j]^2 = 1$, the equality (12) is valid.

To prove the first relation in (13), we note that $[\lambda \cup \tau, \mu, \sigma] = [\lambda, \mu, \sigma][\tau, \mu, \sigma]$, if $\lambda \cap \tau = \emptyset$. Therefore,

$$[\lambda \Delta \tau, \mu, \sigma] = [\lambda \setminus \tau, \mu, \sigma][\lambda \cap \tau, \mu, \sigma][\lambda \cap \tau, \mu, \sigma][\tau \setminus \lambda, \mu, \sigma] = [\lambda, \mu, \sigma][\tau, \mu, \sigma].$$

Analogously, we prove $[\sigma, \lambda \Delta \tau, \mu] = [\sigma, \lambda, \mu][\sigma, \tau, \mu]$ and $[\sigma, \mu, \lambda \Delta \tau] = [\sigma, \mu, \lambda][\sigma, \mu, \tau]$.

We are going to prove (14) and (15) in the cases $\sigma \cap \mu = \sigma \cap \lambda = \mu \cap \lambda = \emptyset$. First we consider $i, k \in \sigma$ and $j \in \lambda$. Suppose $i < k$. We have to analyze three cases: 1) $i < j < k$, 2) $i < k < j$ and 3) $j < i < k$. In the first case, the product $i \wedge j \wedge k$ appears twice in $[\sigma, \lambda, \sigma][\sigma, \sigma, \lambda]$, as a factor of $[\sigma, \lambda, \sigma]$ and as a factor of $[\sigma, \sigma, \lambda]$. In the second case we have a factor of the form $i \wedge k \wedge j.k \wedge i \wedge j$ in $[\sigma, \sigma, \lambda]$ and there are not nontrivial factors in $[\sigma, \lambda, \sigma]$ with i, j, k , while in the third case we have the factor $k \wedge j \wedge i.i \wedge j \wedge k$ in $[\sigma, \lambda, \sigma]$ and there are not nontrivial factors in $[\sigma, \sigma, \lambda]$, with i, j, k . Those cases where $k < i$ are analogous. Therefore, the relation (14) is valid in the particular case $\sigma \cap \lambda = \emptyset$.

Now let i, j, k be elements of σ, μ and λ respectively. We consider the same three cases to prove the relation (15) in the particular case. In any case we will have the factor $i \wedge j \wedge k$ exactly twice in $[\sigma, \lambda, \mu][\mu, \lambda, \sigma][\sigma, \mu, \lambda][\mu, \sigma, \lambda]$. For example, in the first case there are not factors with i, j, k in $[\sigma, \lambda, \mu]$ and $[\mu, \lambda, \sigma]$, and there is one factor $i \wedge j \wedge k$ in $[\sigma, \mu, \lambda]$ and one factor $j \wedge i \wedge k$ in $[\mu, \sigma, \lambda]$.

To prove the relation (14) in general case, we consider again $\sigma = (1234)$ and $\lambda = (1357)$. The term $[\sigma, \lambda, \sigma] = [1324, 1357, 1324]$ can be rewritten as

$$[13, 13, 13][13, 13, 24][13, 57, 13][13, 57, 24][24, 13, 13][24, 13, 24][24, 57, 13][24, 57, 24].$$

In the same way the term $[\sigma, \sigma, \lambda] = [1324, 1324, 1357]$ can be rewritten as

$$[13, 13, 13][13, 13, 57][13, 24, 13][13, 24, 57][24, 13, 13][24, 13, 57][24, 24, 13][24, 24, 57].$$

By definition, $13 \cap 57 = 13 \cap 24 = 24 \cap 57 = \emptyset$. So we obtain the relation (14) in the general case.

Next, by the relation (14) we obtain that $[\sigma \Delta \mu, \lambda, \sigma \Delta \mu][\sigma \Delta \mu, \sigma \Delta \mu, \lambda] = 1$ and by the relation (13) we prove the relation (15) in the general case. Finally, the relation (16) is obtained directly from relations (13), (14) and (15).

□

Theorem 1. *The set \mathcal{F} defined above is a Moufang loop in the variety \mathcal{E} .*

Proof. By the definition of \mathcal{F} we have $\mathcal{F} \in \mathcal{E}$. We consider $\sigma, \mu, \lambda \in V$. We are going to prove the following Moufang identity:

$$(\sigma.\mu).(\lambda.\sigma) = (\sigma.(\mu.\lambda)).\sigma \quad (18)$$

By the Definition (6) we have:

$$(\sigma.\mu).(\lambda.\sigma) = ((\sigma \Delta \mu). \{\sigma, \mu\}. [\sigma, \mu] . [\sigma, \mu, \mu]).((\lambda \Delta \sigma). \{\lambda, \sigma\}. [\lambda, \sigma] . [\lambda, \sigma, \sigma])$$

And by (6) and (7) we have:

$$\begin{aligned}
(\sigma.\mu).(\lambda.\sigma) &= (\mu\Delta\lambda).\{\sigma\Delta\mu, \lambda\Delta\sigma\}.\{\sigma, \mu\}.\{\lambda, \sigma\}. \\
&\quad [\sigma\Delta\mu, \lambda\Delta\sigma]. [\sigma, \mu]. [\lambda, \sigma]. [\sigma, \mu, \mu]. \\
&\quad [\sigma\Delta\mu, \lambda\Delta\sigma, \lambda\Delta\sigma]. [\lambda, \sigma, \sigma]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\sigma.(\mu.\lambda)).\sigma &= (\sigma.(\mu\Delta\lambda).\{\mu, \lambda\}.\{\mu, \lambda\}.\{\mu, \lambda, \lambda\}).\sigma \\
&= (\sigma\Delta(\mu\Delta\lambda).\{\sigma, \mu\Delta\lambda\}.\{\mu, \lambda\}.\{\mu, \lambda\}. \\
&\quad [\sigma, \mu\Delta\lambda]. [\sigma, \mu\Delta\lambda, \mu\Delta\lambda]. [\mu, \lambda, \lambda]).\sigma \\
&= (\mu\Delta\lambda).\{\sigma\Delta\mu\Delta\lambda, \sigma\}.\{\sigma, \mu\Delta\lambda\}.\{\mu, \lambda\}. \\
&\quad [\sigma\Delta\mu\Delta\lambda, \sigma]. [\sigma, \mu\Delta\lambda]. [\mu, \lambda]. [\mu, \lambda, \lambda]. \\
&\quad [\sigma\Delta\mu\Delta\lambda, \sigma, \sigma]. [\sigma, \mu\Delta\lambda, \mu\Delta\lambda]
\end{aligned}$$

Now the equality (18) follows from the Lemma 1. □

We consider a subloop of \mathcal{F} , denoted by \mathcal{F}_n and with generators $\{i \in I_n\}$, where $I_n = \{1, \dots, n\}$.

The next proposition contains important results that we are going to use throughout this paper. The demonstration of the first and second result comes directly from ([4], Theorem 2). The last result is proved easily.

Proposition 1. *Let F be a Moufang loop.*

1. *If $(x, y, z)^2 = 1$ and all the commutators and associators of F are central, then*

$$[xy, z] = [x, z][y, z](x, y, z).$$

2. *If the commutators and associators of F are central, then*

$$(wx, y, z) = (w, y, z)(x, y, z).$$

3. *If the squares and commutators of a Moufang loop F are central, then*

$$(xy)^2 = x^2y^2[x, y].$$

Lemma 2. *Let F_n be a free loop in \mathcal{E} , with free generator set $\{x_1, \dots, x_n\}$.*

Then for all $z \in \mathcal{Z}(F_n)$, there are $\xi_1, \dots, \xi_n, \xi_{ij}, \xi_{ijk} \in \{0, 1\}$, with $i, j, k = 1, \dots, n$ such that

$$z = \prod_{i=1}^n (x_i^2)^{\xi_i} \cdot \prod_{i < j} [x_i, x_j]^{\xi_{ij}} \cdot \prod_{i < j < k} (x_i, x_j, x_k)^{\xi_{ijk}}. \quad (19)$$

Proof. Let \mathcal{N} be a central subloop of F_n generated by the set $\{x_i^2, [x_i, x_j], (x_i, x_j, x_k) | i, j, k \in I_n\}$.

By the Proposition 1 we have:

$$\begin{aligned} (x_i x_j)^2 &= x_i^2 x_j^2 [x_i, x_j]; \\ [x_i x_j, x_k] &= [x_i, x_k] [x_j, x_k] (x_i, x_j, x_k); \\ (x_i x_j, x_k, x_p) &= (x_i, x_k, x_p) (x_j, x_k, x_p). \end{aligned}$$

Hence F_n/\mathcal{N} is an abelian group of exponent two. Then $\mathcal{N} = \mathcal{Z}(F_n)$ and the Lemma is proved. \square

Theorem 2. \mathcal{F}_n is a free loop in the variety \mathcal{E} of rank n .

Proof. Let $F_n \in \mathcal{E}$, $F_n = \langle x_1, \dots, x_n \rangle$ be a free loop with n generators, then by the definition of free loop there is a unique surjective morphism $\varphi : F_n \rightarrow \mathcal{F}_n$ such that $\varphi(x_i) = i$.

To show that \mathcal{F}_n is free, we just need to prove that φ is injective. It is clear that $\ker \varphi \subseteq \mathcal{Z}(F_n)$. By the Lemma 2 any $z \in \ker \varphi \subseteq \mathcal{Z}(F_n)$ has the form given in (19). Then $\varphi(z) \neq 1$ by definition of \mathcal{F} . Therefore, \mathcal{F}_n is free loop. \square

Corollary 1. (1) For any code loop L of rank n there is a homomorphism $\varphi : \mathcal{F}_n \rightarrow L$ such that $\varphi(\mathcal{F}_n) = L$ and $\text{codim}_{\mathcal{Z}(\mathcal{F}_n)} \ker(\varphi) = 1$.

(2) For all \mathbf{F}_2 -subspace $T \subset \mathcal{Z}(\mathcal{F}_n)$ of codimension 1 we have a code loop $L(T) = \mathcal{F}_n/T$.

(3) The loop $L(T)$ is a group if and only if $T \supseteq U_n = (\mathcal{F}_n, \mathcal{F}_n, \mathcal{F}_n) = \mathcal{F}_n \cap U$, where $U = (\mathcal{F}, \mathcal{F}, \mathcal{F})$.

Proposition 2. Let T_1 and T_2 be \mathbf{F}_2 -subspaces of $\mathcal{Z}(\mathcal{F}_n)$ of codimension 1, so $L(T_1) \cong L(T_2)$ if and only if there is an automorphism φ of \mathcal{F}_n such that $T_1^\varphi = T_2$.

Proof. Let $\varphi : \mathcal{F}_n \rightarrow \mathcal{F}_n$ be an automorphism such that $T_1^\varphi = T_2$, where T_1 and T_2 are \mathbf{F}_2 -subspaces of $\mathcal{Z}(\mathcal{F}_n)$ of codimension 1. We define $\bar{\varphi} : \mathcal{F}_n/T_1 \rightarrow \mathcal{F}_n/T_2$ by $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$, for all $\bar{x} \in \mathcal{F}_n/T_1$. Clearly, $\bar{\varphi}$ is an isomorphism between code loops.

Now we construct the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_1 & \xrightarrow{i_1} & \mathcal{F}_n & \xrightarrow{\pi_1} & L_1 \longrightarrow 1 \\ & & \downarrow \varphi = \bar{\sigma}|_{T_1} & & \downarrow \bar{\sigma} & & \downarrow \sigma \\ 1 & \longrightarrow & T_2 & \xrightarrow{i_2} & \mathcal{F}_n & \xrightarrow{\pi_2} & L_2 \longrightarrow 1 \end{array}$$

We suppose that $L_j = \mathcal{F}_n/T_j$, $\sigma : L_1 \rightarrow L_2$ is an isomorphism, π_j is a surjective homomorphism and i_j is the inclusion application, for $j = 1, 2$. Thus,

$$1 \longrightarrow T_j \xrightarrow{i_j} \mathcal{F}_n \xrightarrow{\pi_j} L_j \longrightarrow 1$$

is a short exact sequence, for $j = 1, 2$.

Let x_1, \dots, x_n be a set of free generators of \mathcal{F}_n and $v_i = \sigma(\pi_1(x_i)) \in L_2$. Since π_2 is surjective, there exists $y_i \in \mathcal{F}_n$ such that $\pi_2(y_i) = v_i$. Then there exists a homomorphism $\tilde{\sigma} : \mathcal{F}_n \rightarrow \mathcal{F}_n$ such that $\tilde{\sigma}(x_i) = y_i$.

So, since both the right part of the diagram is commutative by construction, we have $\varphi(T_1) = \tilde{\sigma}|_{T_1}(T_1) \subset T_2$.

We have that both T_1 and T_2 has the same dimension, then we just need to prove that φ is surjective. In fact, let $x \in T_2$, then $i_2(x) = x \in \mathcal{F}_n$, but $\tilde{\sigma}$ is surjective, thus, there exists $i \in \mathcal{F}_n$ such that $\tilde{\sigma}(i) = x$. On the other hand, $x \in \text{Ker}(\pi_2) = \text{Im}(i_2)$, because the sequences of lines of the previous diagram are short exact sequences. Hence $\pi_2(x) = T_2$, that is, $\pi_2(\tilde{\sigma}(i)) = T_2$. As the right part of the diagram is commutative, we have $\sigma(\pi_1(i)) = T_2$. Then $\pi_1(i) \in \text{Ker}(\sigma) = \{T_1\}$, because σ is bijective, that is, $\pi_1(i) = T_1$. Hence, $i \in \text{Ker}(\pi_1) = \text{Im}(i_1) = T_1$, as we wanted. □

We denote by \mathcal{L}_n the set of all subspaces $T \subset \mathcal{Z}(\mathcal{F}_n)$ of codimension 1 such that $T \not\subseteq U_n$. The group $G_n = \text{Aut}\mathcal{F}_n$, group of automorphisms of \mathcal{F}_n , acts on \mathcal{L}_n .

Let L_n be the set of the correspondents G_n -orbits. We denote by O_T the orbit of T , that is, $O_T = \{T^\sigma | \sigma \in G_n\}$.

Corollary 2. *The correspondence $T \longrightarrow L(T)$ gives a bijection between L_n and the set of the isomorphism classes of nonassociative code loops of rank n .*

Let L be a code loop with generator set $X = \{x_1, \dots, x_n\}$ and center $\{1, -1\}$. Then we define the characteristic vector of L , denoted by $\lambda_X(L)$ or $\lambda(L)$, by

$$\lambda(L) = (\lambda_1, \dots, \lambda_n; \lambda_{12}, \dots, \lambda_{1n}, \dots, \lambda_{(n-1)n}; \lambda_{123}, \dots, \lambda_{12n}, \dots, \lambda_{(n-2)(n-1)n}),$$

where $\lambda_i, \lambda_{ij}, \lambda_{ijk} \in \mathbf{F}_2$, $(-1)^{\lambda_i} = x_i^2$, $(-1)^{\lambda_{ij}} = [x_i, x_j]$ e $(-1)^{\lambda_{ijk}} = (x_i, x_j, x_k)$.

With $\lambda(L) \neq 0$ we can associate a subspace $T = T_\lambda$ in $\mathcal{Z}(\mathcal{F}_n)$ of codimension 1 as follows:

We choose an element $\lambda_{ijk} = 1$ and we denote it by $x = x_i \wedge x_j \wedge x_k$. Then $T_\lambda = \mathbf{F}_2 \left\{ x_i^2, x x_j^2 | \lambda_i = 0, \lambda_j = 1 \right\} + \mathbf{F}_2 \left\{ [x_p, x_q], x[x_i, x_j] | \lambda_{pq} = 0, \lambda_{ij} = 1 \right\} + \mathbf{F}_2 \left\{ (x_i, x_j, x_k), x(x_p, x_q, x_l) | \lambda_{ijk} = 0, \lambda_{pql} = 1 \right\}$, by definition.

The following proposition is a simple corollary of definitions.

Proposition 3. *The application $\lambda \longrightarrow T_\lambda$ is a bijection between the set of characteristic vectors of nonassociatives code loops of rank n and the subspaces T in $\mathcal{Z} = \mathcal{Z}(\mathcal{F}_n)$ of codimension 1 such that $U_n \not\subseteq T$.*

So we have a G_n -action over the set of the characteristic vectors. The action is defined by $\lambda^\varphi = \mu$ if and only if $T_\lambda^\varphi = T_\mu$, for every $\varphi \in G_n = \text{Aut}(\mathcal{F}_n)$.

Without loss of generality we can assume $\lambda_{123} = 1$. In this case, we have 2^m characteristic vectors where $m = n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} - 1$. Let $\{O_1, \dots, O_k\}$ be a set of G_n -orbits of characteristic vectors of the code loops of rank n . Then

$$\sum_{i=1}^k |O_i| = 2^m. \quad (20)$$

For a given code loop L , we denote by $\text{Aut}L$ the group of automorphisms of L and we define $\text{Out}L = \text{Aut}L/N(\text{Aut}L)$, the group of outer automorphisms of L . By definition, $N(\text{Aut}L) = \{\phi \in \text{Aut}L \mid \phi(x) = \pm x, \forall x \in L\}$.

Let $G_n^i = \{\varphi \in G_n \mid \lambda^\varphi = \lambda\}$. We note that $\varphi \in G_n^i$ induces an automorphism of the correspondent code loop $L_i = \mathcal{F}_n/T_\lambda$.

For the next proposition, we define $\mathcal{Z}^n = \{\varphi \in G_n \mid \forall i, \varphi(i) = iz_i, z_i \in \mathcal{Z}(\mathcal{F}_n)\} \simeq \mathcal{Z}(\mathcal{F}_n)^n$ and we denote by $GL_n(2)$ the General Linear Group of degree n over a finite field with 2 elements.

Proposition 4. *Let $\overline{GL_n(2)} = \{\varphi \in G_n \mid i^\varphi \subseteq V, \text{ for all } i \in \{1, \dots, n\}\} \subset G_n$. Then:*

1. $G_n = \overline{GL_n(2)} \cdot \mathcal{Z}^n$;
2. $G_n/\mathcal{Z}^n \simeq GL_n(2) = \text{Aut}_{\mathbf{F}_2} V$ and
3. $G_n^i/\mathcal{Z}^n \simeq \text{Out}(L_i)$.

Proof. Let $\varphi \in G_n$, so $\varphi(i) = \sigma_i \cdot z_i$, for $\sigma_i \in V = \{1, \dots, n\}$ and $z_i \in \mathcal{Z}$. Thus, for all $\varphi \in G_n$, there are $\{\sigma_1, \dots, \sigma_n\}$ a corresponding basis of V and $(z_1, \dots, z_n) \in \mathcal{Z}^n$. We have that $\mathcal{F}_n/\mathcal{Z}(\mathcal{F}_n) \simeq V = \{1, \dots, n\}$ is a vector \mathbf{F}_2 -space and, for $\varphi \in G_n$, we have an automorphism $\tilde{\varphi} : \mathcal{F}_n/\mathcal{Z}(\mathcal{F}_n) \longrightarrow \mathcal{F}_n/\mathcal{Z}(\mathcal{F}_n)$ such that $i^{\tilde{\varphi}} = \sigma_i$. Then $\tilde{\varphi} \in GL(V) = GL_n(2)$.

We are going to show now that $G_n^i/\mathcal{Z}^n \simeq \text{Out}(L_i)$. In fact, let $\varphi \in G_n^i$, then $T_\lambda^\varphi = T_\lambda$, hence $\tilde{\varphi} : \mathcal{F}_n/T_\lambda \longrightarrow \mathcal{F}_n/T_\lambda$ is an isomorphism of code loops. The homomorphism $\pi_i : G_n^i \longrightarrow \text{Aut}(L_i)$ such that $\pi_i(\varphi) = \tilde{\varphi}$, is surjective (by the Proposition 2) with kernel $K_i = \{\tau \in G_n^i \mid \tau(x_j) = x_j v_j, v_j \in T_\lambda\}$.

Consider also the surjective homomorphism $\psi_i : \text{Aut}(L_i) \longrightarrow \text{Out}(L_i)$. Then $\pi = \psi_i \circ \pi_i : G_n^i \longrightarrow \text{Out}(L_i)$ is a surjective homomorphism. We have $\mathcal{Z}^n \subseteq \ker(\pi)$. Now, let $\varphi \in \ker(\pi)$. If $\varphi \notin \mathcal{Z}^n$, there is $i^\varphi \neq iz_i$, for all $z_i \in \mathcal{Z}$, then for $\pi(\varphi) \in \text{Out}(L_i)$, we have $\pi(\varphi) : i \longmapsto \pm i^\varphi \neq \pm i \pmod{T_\lambda}$, hence $\pi\varphi \neq 1$ in $\text{Out}(L_i)$. Thus, $\ker(\pi) = \mathcal{Z}^n$ and, therefore, $G_n^i/\mathcal{Z}^n \simeq \text{Out}(L_i)$. \square

We denote G_n^i/\mathcal{Z}^n by $\text{Stab}(\lambda)$.

Definition 2. A 3-space is a \mathbf{F}_2 -space V with an antisymmetric trilinear form $(\ , \ , \)$ with values in \mathbf{F}_2 .

A 2-space is a 3-space V with a bilinear map $[\ , \] : V \times V \longrightarrow \mathbf{F}_2$ such that $[v, wu] = [wu, v] = [v, w][v, u](v, w, u)$.

For a given code loop L , we can associate a 2-space $V = L/\mathcal{Z}(L)$ with a 3-form induced by the associator and the correspondent antisymmetric application induced by the commutator.

Let V be a 3-space. By definition, the nucleus of V is the subspace $N_3(V) = \{v \in V | (v, V, V) = 1\}$. Similarly, for a 2-space V , we define $N_2(V) = \{v \in V | (v, V, V) = [v, V] = 1\}$.

Definition 3. Let L_1 and L_2 be code loops. We define the product of L_1 and L_2 as a code loop $L_1 * L_2 = \frac{L_1 \times L_2}{D}$, where $D = \{1 \times 1, (-1) \times (-1)\}$ is a diagonal of the center of $L_1 \times L_2$.

Let L be an other code loop, we will write $L_1 \sim_L L_2$ if $L * L_1 \simeq L * L_2$ and we will say that L_1 and L_2 are L -equivalents.

For the following, \mathbb{Z}_m is the (additive) group of integers modulo m .

Lemma 3. Two code loops L_1 and L_2 are isomorphic as 2-spaces if and only if $L_1 \sim_{\mathbb{Z}_4} L_2$.

Proof. Let L_1 and L_2 be \mathbb{Z}_4 -equivalents. Therefore, there exists an isomorphism $\sigma : \mathbb{Z}_4 * L_1 \longrightarrow \mathbb{Z}_4 * L_2$. We define a linear map $\tau : L_1 \longrightarrow L_2$ by $\tau(v) = w \in L_2$ if $\sigma(v) = aw$, where $a \in \mathbb{Z}_4$. Since by definition $[a, L_2] = (a, L_2, L_2) = 1$, then τ is an isomorphism of L_1 and L_2 as 2-spaces.

Conversely, if $\tau : L_1 \longrightarrow L_2$ is an isomorphism of code loops as 2-spaces, so the application $\sigma : \mathbb{Z}_4 * L_1 \longrightarrow \mathbb{Z}_4 * L_2$, where $\sigma(a \times v) = b \times \tau(v)$ and $b = a$ if $v^2 = \tau(v)^2$, $b = ac$ if $v^2 \neq \tau(v)^2$, $a, b, c \in \mathbb{Z}_4$, $c^2 = -1$, is an isomorphism. \square

3 Automorphisms of Code Loops

Classification of all code loops until rank 4 using the definition of characteristic vector has been done in [1]. In this section we will illustrate the notions introduced in this case.

3.1 Code loops of rank 3

Let L be a nonassociative code loop of rank 3 with generators a, b, c . Then $(a, b, c) = -1$. We associate to L the characteristic vector $\lambda(L)$, defined by $\lambda(L) = (\lambda_1, \dots, \lambda_6)$ where $a^2 = (-1)^{\lambda_1}, b^2 = (-1)^{\lambda_2}, c^2 = (-1)^{\lambda_3}, [a, b] = (-1)^{\lambda_4}, [a, c] = (-1)^{\lambda_5}, [b, c] = (-1)^{\lambda_6}$.

Theorem 3. Consider C_1^3, \dots, C_5^3 the code loops with the following characteristic vectors:

$$\begin{aligned}\lambda(C_1^3) &= (1, 1, 1, 1, 1, 1), & \lambda(C_2^3) &= (0, 0, 0, 0, 0, 0), & \lambda(C_3^3) &= (0, 0, 0, 1, 1, 1), \\ \lambda(C_4^3) &= (1, 1, 0, 0, 0, 0), & \lambda(C_5^3) &= (1, 0, 0, 0, 0, 0).\end{aligned}$$

Then any two loops from the list $\{C_1^3, \dots, C_5^3\}$ are not isomorphic and all nonassociative code loop of rank 3 is isomorphic to one of this list.

Proof. Let a, b, c be the generators of the nonassociative code loop L of rank 3. Thus $(a, b, c) = -1$. There are two possibilities:

1. $[a, b] = 1$;
2. $[a, b] = [a, c] = [b, c] = -1$.

In the first case, we can choose generators such that $[a, b] = [a, c] = [b, c] = 1$. In fact, if $[a, c] = -1$, we can take $x = bc$ and we will have $[a, x] = [a, b][a, c](a, b, c) = 1$, (see Theorem 2, [4]).

If $[a, b] = [a, c] = 1$ and since $[b, c] = -1$, we obtain $[a, ab] = [a, c] = [ab, c] = 1$.

Fixed the generators a, b, c such that $[a, b] = [a, c] = [b, c] = 1$, we will have for $\lambda(L)$ four possibilities: $(0, 0, 0, 0, 0, 0), (1, 1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0), (1, 1, 1, 0, 0, 0)$.

Notice that the loops that correspond to the last two characteristic vectors are isomorphics. In fact, let L_1 be the code loop with generator set $X = \{a, b, c\}$ and with corresponding characteristic vector $\lambda_X(L_1) = (1, 1, 1, 0, 0, 0)$. Then, for the generator set $Y = \{a, ab, ac\}$, we have $a^2 = -1, (ab)^2 = 1, (ac)^2 = 1$, that is, $\lambda_Y(L_1) = (1, 0, 0, 0, 0, 0)$.

In the second case, there are two possibilities:

- i) $a^2 = b^2 = c^2 = -1$: Here we have $\lambda(L) = (1, 1, 1, 1, 1, 1)$.
- ii) $a^2 = 1$: If we assume $b^2 = -1$ or $c^2 = -1$, we will have $(ac)^2 = (ab)^2 = 1$. Hence we can choose generators a, b, c such that $a^2 = b^2 = c^2 = 1$.
Therefore, $\lambda(L) = (0, 0, 0, 1, 1, 1)$.

□

Proposition 5. Let O_1^3, \dots, O_5^3 be the orbits of the characteristic vectors associated to code loops C_1^3, \dots, C_5^3 , respectively. Then

$$\begin{aligned}O_1^3 &= \{(111111)\}; \\ O_2^3 &= \{(000000), (000001), (000010), (000100), (001011), (010101), (100110)\}; \\ O_3^3 &= \{(000111), (001111), (010111), (011111), (100111), (101111), (110111)\};\end{aligned}$$

$$\begin{aligned}
O_4^3 = & \{(001100), (010010), (011000), (011001), (011011), (011101), (100001), \\
& (101000), (101010), (101011), (101110), (110000), (110100), (110101), \\
& (110110), (111001), (111010), (111011), (111100), (111101), (111110)\}; \\
O_5^3 = & \{(000011), (000101), (000110), (001000), (001001), (001010), (001101), \\
& (001110), (010000), (010001), (010011), (010100), (010110), (011010), \\
& (011100), (011110), (100000), (100010), (100011), (100100), (100101), \\
& (101001), (101100), (101101), (110001), (110010), (110011), (111000)\}.
\end{aligned}$$

We note that $|O_1^3| = 1, |O_2^3| = |O_3^3| = 7, |O_4^3| = 21, |O_5^3| = 28$ and $\sum_{i=1}^5 |O_i^3| = 64 = 2^6$. Therefore, $|\text{Out}C_1^3| = |GL_3(2)| = 168$, $|\text{Out}C_2^3| = |\text{Out}C_3^3| = 24$, $|\text{Out}C_4^3| = 8$ and $|\text{Out}C_5^3| = 6$.

Proof. Let (a, b, c) be a fixed set of generators for each code loop C_i^3 , $i = 1, \dots, 5$. We consider the characteristic vector $\lambda(C_i^3)$ as representative of each orbit O_i^3 , $i = 1, \dots, 5$. To find all characteristic vectors belonging to the orbit of each representative, we have to find the different set of generators formed from the set of generators (a, b, c) fixed. In the Table 1, we present all the values of squares and commutators obtained from each characteristic vector, that is, the squares of each generator element and the commutator between each pair of generators elements. Each column i corresponds to a code loop C_i^3 , $i = 1, \dots, 5$.

Table 1: Squares and Commutators from fixed generators a, b, c for each C_i^3

	1	2	3	4	5
a^2	-1	1	1	-1	-1
b^2	-1	1	1	-1	1
c^2	-1	1	1	1	1
$[a, b]$	-1	1	-1	1	1
$[a, c]$	-1	1	-1	1	1
$[b, c]$	-1	1	-1	1	1

Let $\varphi \in GL_3(2)$ an automorphism of C_i^3 . Then we find a new set of generators of C_i^3 , denoted by (x, y, z) , that is, $(a^\varphi = x, b^\varphi = y, c^\varphi = z)$. We see in the Table 2 all the possibilities for x, y and z , and the squares of x, y, z and the commutators between each pair of elements.

In the first orbit we have only the vector $\lambda(C_1^3) = (111111)$, because for any x, y, z as above, we always have $x^2 = y^2 = z^2 = -1$ and $[x, y] = [x, z] = [y, z] = -1$.

To determine the vectors of the orbit O_2^3 , we observe from the Table 2 that the set

Table 2: Squares and Commutators of x, y and z

	1	2	3	4	5		1	2	3	4	5
$(ab)^2$	-1	1	-1	1	-1	$[b, abc]$	-1	-1	-1	-1	-1
$(ac)^2$	-1	1	-1	-1	-1	$[c, ab]$	-1	-1	-1	-1	-1
$(bc)^2$	-1	1	-1	-1	1	$[c, ac]$	-1	1	-1	1	1
$(abc)^2$	-1	-1	1	-1	1	$[c, bc]$	-1	-1	1	1	1
$[a, ab]$	-1	1	-1	1	1	$[c, abc]$	-1	1	-1	-1	-1
$[a, ac]$	-1	1	-1	1	1	$[ab, ac]$	-1	1	-1	1	1
$[a, bc]$	-1	-1	-1	-1	-1	$[ab, bc]$	-1	1	-1	1	1
$[a, abc]$	-1	-1	-1	-1	-1	$[ab, abc]$	-1	-1	-1	-1	-1
$[b, ab]$	-1	1	-1	1	1	$[ac, bc]$	-1	1	1	1	1
$[b, ac]$	-1	-1	-1	-1	-1	$[ac, abc]$	-1	-1	-1	-1	-1
$[b, bc]$	-1	1	-1	1	1	$[bc, abc]$	-1	-1	-1	-1	-1

of generators has the following possibilities: $(x, y, z) = (a, b, c), (a, b, ac), (a, b, bc), (a, bc, b), (abc, ac, bc), (ac, abc, bc)$ and (ac, bc, abc) , which corresponds, respectively, to the characteristic vectors $(000000), (000001), (000010), (000100), (100110), (010101)$ and (001011) . All these vectors form the orbit O_2^3 . In the same way, we obtain the vectors of O_3^3 . The possibilities for (x, y, z) , in this case, are $(a, b, c), (ab, a, c), (a, ab, c), (a, c, ab), (ab, ac, a), (ab, a, ac)$ and (a, ab, ac) , with characteristic vectors respectively given by $(000111), (100111), (010111), (001111), (110111), (101111)$ and (011111) . Analogously, we obtain all the vectors of the orbits O_4^3 and O_5^3 .

□

For the following statements we use the notation $GR\{x_1, \dots, x_n\}$ to denote the group generated by the elements x_1, \dots, x_n . Here, D_n denotes the dihedral group of order n and S_n is the symmetric group on n letters.

Proposition 6. *In the notation above, we have:*

(1) $\text{Out}C_1^3 \simeq GL_3(2)$

(2) $\text{Out}C_2^3 \simeq S_4$

(3) $\text{Out}C_3^3 \simeq S_4$

(4) $\text{Out}C_4^3 \simeq D_8$

(5) $\text{Out}C_5^3 \simeq S_3$

Proof. 1. Let a, b, c be generators of $L = C_1^3$ with corresponding characteristic vector $\lambda = \lambda(L) = (111111)$, that is $a^2 = b^2 = c^2 = -1$ and $[a, b] = [a, c] = [b, c] = -1$. From the Tables 1 and 2 we have $(a^\varphi)^2 = (b^\varphi)^2 = (c^\varphi)^2 = -1$ and $[a^\varphi, b^\varphi] = [a^\varphi, c^\varphi] = [b^\varphi, c^\varphi] = -1$, for all $\varphi \in GL_3(2)$.

2. We want to determine all the outer automorphisms of the code loop $L = C_2^3$. We consider $\lambda = \lambda(L) = (000000)$ as a representative characteristic vector of L of the orbit O_2^3 and (a, b, c) a set of generators for L such that $a^2 = b^2 = c^2 = 1$ and $[a, b] = [a, c] = [b, c] = 1$.

We consider $\varphi_1, \dots, \varphi_6 \in GL_3(2)$ such that

$$\begin{aligned} \varphi_1 &= id_{3 \times 3} & \varphi_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \varphi_3 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \varphi_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \varphi_5 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \varphi_6 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

We have $\lambda^{\varphi_i} = \lambda$, for each $i = 1, \dots, 6$. Then $\varphi_i \in \text{Stab}(\lambda)$, for each $i = 1, \dots, 6$. We observe that $GR\{\varphi_1, \dots, \varphi_6\} = S_3$. Therefore, $S_3 \subseteq \text{Stab}(\lambda)$. From the Proposition 5, in O_2^3 there are exactly 7 characteristic vectors, so we have to prove that there are

24 automorphisms fixing λ . Let $\sigma \in GL_3(2)$ defined by $\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We have

$\lambda^\sigma = \lambda$, hence $\sigma \in \text{Stab}(\lambda)$. Besides, $\sigma^3 = 1$ and $\sigma^2 \in \text{Stab}(\lambda)$. Indeed, for all $i = 1, \dots, 6$ we have $(\sigma\varphi_i), (\sigma^2\varphi_i), (\varphi_i\sigma), (\varphi_i\sigma^2) \in \text{Stab}(\lambda)$. From these elements we obtain more 16 different stabilizers (automorphisms that fix λ). For complete our group we consider $(\sigma\varphi_3)\sigma^2$ and $(\sigma^2\varphi_5)\sigma^2$.

We know that $\text{Out } C_2^3 \simeq \text{Stab}(\lambda)$. We must prove $\text{Stab}(\lambda) = S_4$. In fact, we first denote $\varphi_2, \varphi_3, (\sigma\varphi_3)\sigma^2$, respectively, by $\sigma_1, \sigma_2, \sigma_3$ and after, with direct calculations we prove that $\varphi_2^2 = \varphi_3^2 = (\sigma\varphi_3)\sigma^2 = id$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$, $\sigma_1\sigma_3 = \sigma_3\sigma_1$. Therefore, $\text{Stab}(\lambda) = GR\{\sigma_1, \sigma_2, \sigma_3 | \sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, j \neq i \pm 1, i = 1, 2, 3\} = S_4$.

3. Let a, b, c be generators of the code $L = C_3^3$ with characteristic vector $\lambda = \lambda(L) = (000111)$. We have that $\text{Stab}(\lambda)$ is the group of all permutations of the set $X = \{a, b, c, abc\}$. We just consider $\varphi \in GL_3(2)$ such that $\varphi(a) = x$, $\varphi(b) = y$ and $\varphi(c) = z$. We have to find (x, y, z) such that $x^2 = y^2 = z^2 = 1$ and $[x, y] = [x, z] = [y, z] = -1$. From the Tables 1 and 2, we obtain all possible values of x, y, z

satisfying the relations desired. Thus, $x, y, z \in \{a, b, c, abc\}$. We consider $\sigma_i \in GL_3(2)$, for $i = 1, 2, 3$, such that $\sigma_1(a) = a, \sigma_1(b) = b, \sigma_1(c) = abc; \sigma_2(a) = a, \sigma_2(b) = abc, \sigma_2(c) = c; \sigma_3(a) = b, \sigma_3(b) = a, \sigma_3(c) = c$.

Analogously to the previous case, we see that $\text{Stab}(\lambda) = GR\{\sigma_1, \sigma_2, \sigma_3 | \sigma_i^2 = 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i, j \neq i \pm 1, i = 1, 2, 3\} = S_4$.

4. Let a, b, c be generators of the code loop $L = C_4^3$ such that $\lambda = \lambda(L) = (110000)$. Since there are 21 characteristic vectors in O_4 , we have 8 automorphisms fixing λ . To prove that $\text{Out } L \simeq D_8$, we must first find two generators, φ and σ , φ of order 4 and σ of order 2, fixing λ , and such that $\sigma\varphi\sigma = \varphi^3$. We consider $\varphi \in GL_3(2)$ and $\sigma \in GL_3(2)$ such that $\varphi(a) = b, \varphi(b) = bc, \varphi(c) = ab; \sigma(a) = b, \sigma(b) = a$ and $\sigma(c) = c$. We have that φ and σ fix λ . Besides, φ has order 4, σ has order 2 and the relations $\sigma\varphi^3 = \varphi\sigma, \varphi^2\sigma = \sigma\varphi^2$ and $\varphi^3\sigma = \sigma\varphi$ are valid.

Therefore, $\text{Out } C_4^3 \simeq D_8 = GR\{\sigma, \varphi | \varphi^4 = 1, \sigma^2 = 1, \sigma\varphi\sigma^{-1} = \varphi^{-1}\} = \text{Stab}(\lambda)$.

5. Let a, b, c be generators of code loop $L = C_5^3$ such that $\lambda = \lambda(L) = (100000)$. Since we have 28 different characteristic vectors in the orbit O_5 , we have to find 6 automorphisms fixing λ . Let $\varphi \in GL_3(2)$ and $\sigma \in GL_3(2)$ such that $\varphi(a) = a, \varphi(b) = c, \varphi(c) = b; \sigma(a) = ac, \sigma(b) = bc$ and $\sigma(c) = c$. We have φ and σ fix $\lambda, \sigma^2 = \varphi^2 = 1$ and $\varphi\sigma\varphi = \sigma\varphi\sigma$. Therefore, $\text{Out } C_5^3 \simeq S_3 = GR\{\varphi, \sigma | \varphi\sigma\varphi = \sigma\varphi\sigma\} = \text{Stab}(\lambda)$.

□

3.2 Code loops of rank 4

Lemma 4. *Let V be a nontrivial 3-space of dimension 4. Then $\dim N(V) = 1$ and V has a base $\{a, b, c, d\}$ such that $(a, b, c) = 1$ and $N(V) = \mathbf{F}_2 d = \{0, d\}$.*

Proof. Since V is a nontrivial 3-space, let $(v, w, u) = 1$, for some $v, w, u \in V$. So there is $t \in V$ such that $(v, w, t) = 0$ and $\{v, w, u, t\}$ is a basis of V . If $(v, w, t) = 1$, then $(v, w, u + t) = (v, w, u) + (v, w, t) = 0$, soon we can assume $(v, w, t) = 0$.

If $(v, u, t) = 1$, then $(v, w, t + w) = (v, u, t + w) = 0$ and we can assume that $(v, u, t) = 0$. If $(w, u, t) = 1$, then $t + v \in N(V)$, thus we can suppose $(w, u, t) = 0$.

Hence, we determine a basis $\{v, w, u, t\}$ such that $(v, w, u) = 1, (v, w, t) = (v, u, t) = (w, u, t) = 0$.

□

Suppose that $X = \{a, b, c, d\}$ is a basis of a nonassociative code loop L of rank 4 such that $N(L) = \mathbf{F}_2 d$. Thus, L has only one nontrivial associator $(a, b, c) = -1$. In this case, a

characteristic vector of L is $\lambda_X = (\lambda_1, \dots, \lambda_{10})$, where

$$\begin{aligned} a^2 &= (-1)^{\lambda_1}, b^2 = (-1)^{\lambda_2}, c^2 = (-1)^{\lambda_3}, d^2 = (-1)^{\lambda_4}, \\ [a, b] &= (-1)^{\lambda_5}, [a, c] = (-1)^{\lambda_6}, [a, d] = (-1)^{\lambda_7}, \\ [b, c] &= (-1)^{\lambda_8}, [b, d] = (-1)^{\lambda_9}, [c, d] = (-1)^{\lambda_{10}}. \end{aligned}$$

Theorem 4. Consider C_1^4, \dots, C_{16}^4 the code loops with the following characteristic vectors. All nonassociative code loop of rank 4 is isomorphic to one from the list. Moreover, none of those loops are isomorphic to each other.

L	$\lambda(L)$	L	$\lambda(L)$
C_1^4	(1110110100)	C_9^4	(0100001000)
C_2^4	(0000000000)	C_{10}^4	(0001111000)
C_3^4	(0000110100)	C_{11}^4	(0001001000)
C_4^4	(0010100000)	C_{12}^4	(0000001100)
C_5^4	(0000010100)	C_{13}^4	(0110111100)
C_6^4	(1111110100)	C_{14}^4	(0001001100)
C_7^4	(0001000000)	C_{15}^4	(1001001100)
C_8^4	(0000001000)	C_{16}^4	(0001111100)

Moreover, we have, if we denote by O_i^4 the orbit of the characteristic vector of the code loop C_i^4 , then $|O_1^4| = 1$, $|O_2^4| = 7$, $|O_4^4| = 21$, $|O_5^4| = |O_{14}^4| = |O_{15}^4| = 28$, $|O_6^4| = 8$, $|O_7^4| = |O_8^4| = |O_{13}^4| = 56$, $|O_9^4| = |O_{12}^4| = |O_{16}^4| = 168$, $|O_{10}^4| = |O_{11}^4| = 112$.

Proof. Let L be a nonassociative code loop of rank 4 with generators a, b, c, d . We can assume $(d, L, L) = 1$. Thus, L has only one nontrivial associator $(a, b, c) = -1$. In this case, a characteristic vector of L is $\lambda_X = (\lambda_1, \dots, \lambda_{10})$, where

$$\begin{aligned} a^2 &= (-1)^{\lambda_1}, b^2 = (-1)^{\lambda_2}, c^2 = (-1)^{\lambda_3}, d^2 = (-1)^{\lambda_4}, \\ [a, b] &= (-1)^{\lambda_5}, [a, c] = (-1)^{\lambda_6}, [a, d] = (-1)^{\lambda_7}, \\ [b, c] &= (-1)^{\lambda_8}, [b, d] = (-1)^{\lambda_9}, [c, d] = (-1)^{\lambda_{10}}. \end{aligned}$$

Let $C = C(d)$ be the centralizer of d , that is, $C = \{v \in L \mid [v, d] = 1\}$.

A subloop L_0 of L is a characteristic subloop if and only if $L_0^\varphi = L_0$, for all $\varphi \in \text{Aut}(L)$. Then, we have that C is a characteristic subloop of L .

If $C = L$, then we have two possibilities:

1. $d^2 = 1$

In this case, $L \simeq \mathbb{Z}_2 \times L_1$, where L_1 is a nonassociative code loop of rank 3. Since there are 5 nonassociative code loops of rank 3, then we obtain 5 non-isomorphic code loops of rank 4, denoted by C_i^4 , that is, $C_i^4 \cong \mathbb{Z}_2 \times C_i^3, i = 1, \dots, 5$.

2. $d^2 = -1$

Let $\langle d \rangle = \{1, -1, d, -d\}$ be the group generated by d . We know that $\langle d \rangle \cong \mathbb{Z}_4$ and that \mathbb{Z}_4 is a code loop. Again, let L_1 be a nonassociative code loop of rank 3. We have that $\mathbb{Z}_4 * L_1$ is a nonassociative code loop of rank 4.

In this case, $L \simeq \mathbb{Z}_4 * L_1$. But, by the Lemma 3, it is possible to prove that $\mathbb{Z}_4 * C_1^3 \cong \mathbb{Z}_4 * C_3^3$ and $\mathbb{Z}_4 * C_2^3 \cong \mathbb{Z}_4 * C_4^3 \cong \mathbb{Z}_4 * C_5^3$. For this, we just prove that C_1^3 and C_3^3 are isomorphic as 2-spaces, as well as C_2^3, C_4^3 and C_5^3 . In fact, a characteristic vector from corresponding 2-space of a given code loop L of rank n can be obtained from characteristic vector $\lambda(L)$ by omitting the first n coordinates of $\lambda(L)$. Therefore, we have 2 new loops: $C_6^4 = \mathbb{Z}_4 * C_1^3$ and $C_7^4 = \mathbb{Z}_4 * C_2^3$.

If $C \neq L$, then C is a code group of rank 3. In fact, we suppose that for all $u \neq d \in X = \{a, b, c, d\}$, the set of generators of L , we have $[u, d] = -1$, but since $[ab, d] = [ac, d] = 1$, hence we have that $\{d, ab, ac\}$ generates C . Now, if we assume that, for example, $[a, d] = 1, [b, d] = [c, d] = -1$, we will have $[bc, d] = 1$, hence $\{d, a, bc\}$ generates C .

There are 5 non-isomorphic code groups of rank 3:

1. $G_1^3 = \mathbb{Z}_2^4$;
2. $G_2^3 = \mathbb{Z}_4 \times \mathbb{Z}_2^2$;
3. $G_3^3 = D_8 \times \mathbb{Z}_2$;
4. $G_4^3 = Q \times \mathbb{Z}_2$, where Q is the Quaternion group;
5. $G_5^3 = D_8 * \mathbb{Z}_4$, where D_8 is the dihedral group with 8 elements.

We are going to do an analysis for each case:

1. $C = G_1^3$. In this case, $b, c, d \in C$, then $b^2 = c^2 = d^2 = [b, c] = [b, d] = [c, d] = 1$ and $[a, d] = -1$. If $a^2 = -1$, then $(ad)^2 = a^2 d^2 [a, d] = 1$. Hence, we can assume $a^2 = 1$. We denote this loop by C_8^4 .
2. $C = G_2^3$. There are 2 cases:
 - (a) $d^2 = 1$. Here, $b^2 = -1, c^2 = 1, [a, d] = -1, [a, b] = [a, c] = 1$. As in the previous case, we can assume $a^2 = 1$. We denote this loop by C_9^4 .

- (b) $d^2 = -1$. We have $b^2 = c^2 = 1$, $[a, d] = -1$. If $[a, b] = [a, c] = -1$, then $[a, bc] = -1$, $(ab)^2 = -a^2$ and hence, we can assume $a^2 = 1$. We denote this loop by C_{10}^4 . If $[a, b] = 1$, then without loss of generality, $[a, c] = 1$, since that $[a, bc] = -[a, c]$. Since $(ac)^2 = -a^2$, we can assume $a^2 = 1$. We denote this loop by C_{11}^4 .
3. $C = G_3^3$. In this case, $d^2 = 1$, $c^2 = b^2 = 1$, $[b, c] = -1$, $[a, d] = -1$. Since $[a, bd] = -[a, b]$, $(ad)^2 = -a^2$, we can assume $[a, b] = [a, c] = 1$ and $a^2 = 1$. We denote this loop by C_{12}^4 .
4. $C = G_4^3$. In this case, $d^2 = 1$, $a^2 = 1$, $b^2 = c^2 = [b, c] = -1$, $[a, d] = -1$. There are 2 possibilities. If $[a, b] = [a, c] = -1$, we denote this loop by C_{13}^4 .
If $[a, b] = [a, c] = 1$, this loop is isomorphic to C_{13}^4 , since we can replace b by bc and c by cd , because $(bc)^2 = (cd)^2 = [bc, cd] = -1$, $[a, bc] = [a, cd] = -1$.
5. $C = G_5^3$. In this case, $d^2 = -1$, $a^2 = 1$, $b^2 = c^2 = 1$, $[b, c] = -1$, $[a, d] = -1$. There are 3 possibilities:
- (a) $[a, b] = [a, c] = 1$, $[a, bc] = -1$. In this subcase, we have two non-isomorphic loops. If $a^2 = 1$, we have the loop denoted by C_{14}^4 , and if $a^2 = -1$ then we have the other loop denoted by C_{15}^4 .
- (b) $[a, b] = [a, c] = -1$, $[a, bc] = -1$. In this other subcase, $(abd)^2 = -a^2$ and $[abd, b] = [abd, c] = -1$. Therefore, we can assume $a^2 = 1$. We denote this loop by C_{16}^4 .
- (c) $[a, b] = 1$, $[a, c] = -1$, $[a, bc] = 1$. This loop is isomorphic to loop C_{16}^4 , since we can replace b by bcd .

□

Corollary 3. *There are five \mathbb{Z}_4 -equivalence classes of code loops of rank 4:*

$$\begin{aligned} Z_1^4 &= \{C_1^4, C_3^4, C_4^4, C_6^4\} & Z_2^4 &= \{C_5^4, C_8^4, C_9^4, C_{10}^4, C_{11}^4\} \\ Z_3^4 &= \{C_{12}^4, C_{14}^4, C_{15}^4\} & Z_4^4 &= \{C_2^4, C_7^4\} \\ Z_5^4 &= \{C_{13}^4, C_{16}^4\} \end{aligned}$$

Proof. From the Lemma 3, we have to prove that $C_i^4 \cong C_j^4$ as 2-spaces if and only if $C_i^4, C_j^4 \in Z_k^4$ for some $k = 1, \dots, 4$. We will write $i \sim j$ instead $C_i^4 \sim_{\mathbb{Z}_4} C_j^4$.

A characteristic vector of a 2-space which corresponds to a given code loop L of rank n , can be obtained from the characteristic vector $\lambda(L)$ by omitting the first n coordinates of $\lambda(L)$.

Therefore, from the Theorem 4, we have $1 \sim 3 \sim 6$, $2 \sim 7$, $8 \sim 9 \sim 11$, $12 \sim 14 \sim 15$ and $13 \sim 16$.

Let $X = \{a, b, c, d\}$ be a set of generators of C_3^4 such that $(a, b, c) = [a, b] = [a, c] = [b, c] = -1$, but then $[a, abc] = [b, abc] = 1$ and, for a set of generators $Y = \{a, b, abc, d\}$, we

have a characteristic vector from the 2-space C_3^4 of the form: $\lambda_Y(C_3^4) = (100000) = \lambda(C_4^4)$. Therefore, $3 \sim 4$.

Let $X = \{a, b, c, d\}$ be a set of generators of C_5^4 such that $(a, b, c) = [a, c] = [a, d] = -1$, but then $[a, cd] = 1$ and for a set of generators $Y = \{a, b, cd, d\}$ we have a characteristic vector of the 2-space C_5^4 of the form: $\lambda_Y(C_5^4) = (001000) = \lambda(C_8^4)$. Therefore, $5 \sim 8$.

Let $X = \{a, b, c, d\}$ a set of generators of C_{10}^4 such that $(a, b, c) = [a, b] = [a, c] = [a, d] = -1$, but then $[a, bd] = [a, cd] = 1$ and, for a set of generators $Y = \{a, bd, cd, d\}$, we have a characteristic vector of the 2-space C_{10}^4 of the form: $\lambda_Y(C_{10}^4) = (001000) = \lambda(C_8^4)$. Therefore, $10 \sim 8$.

Now, we have to prove that all these classes Z_1^4, \dots, Z_5^4 are nonequivalent. A space $C(N(L)) = \{x \in L | [x, N(L)] = 1\}$ is a characteristic space of L . In our case, $N(L) = \mathbf{F}_2 d$ and for $C = C(d)$ we have three possibilities:

1. $C = L$,
2. $[C, C] = (C, C, C) = 1$,
3. $C \neq L$, $[C, C] \neq 1$.

Besides, we have the second possibility only for Z_3^4 and the third possibility only for Z_4^4 . To prove that $1 \not\sim 2$ we note that $N_2(C_1^4) \simeq N_2(C_2^4) \simeq \mathbf{F}_2 d$, but the 2-space $C_1^4/N_2(C_1^4)$ and $C_2^4/N_2(C_2^4)$ are non-isomorphic. Analogously, we prove that $1 \not\sim 5$ and $2 \not\sim 5$.

□

Let (a, b, c, d) be a set of generators of the code loop of rank 4 denoted by C_i^4 and $\sigma \in GL_4(2)$ an automorphism of C_i^4 . For each σ we find a new set of generators of the same code loop, denoted by (u, v, w, d) , that is, $a^\sigma = u, b^\sigma = v, c^\sigma = w, d^\sigma = d$. Analogously to the study of the orbits of the code loops of rank 3, Proposition 5, we can exhibit all the codes of each orbit associated to a given code loop of code 4. In fact, just fix a representative characteristic vector of a certain orbit, runs through $GL_4(2)$ and find all possible set of generators (u, v, w, d) . So from the square of each generator and the commutators of each pair of generators of C_i^4 , we find a characteristic vector that belongs to orbit of the representative characteristic vector fixed.

When is necessary we denote the characteristic vector $\lambda(C_i^k)$ by λ_i^k , where k represents the rank of the code loop. We denote by $\text{Stab}(\lambda_i^k)$ the group of automorphisms of $GL_k(2)$ that fix λ_i^k .

For the next proposition, $H \times K$ means the direct product of the groups H and K , and $H \rtimes K$ means the semidirect product of the groups H and K .

Proposition 7. *In the notation above, we have:*

$$\begin{array}{ll}
\text{Out}C_1^4 \simeq \mathbb{Z}_2^3 \rtimes GL_3(2) & \text{Out}C_9^4 \simeq D_8 \\
\text{Out}C_2^4 \simeq \mathbb{Z}_2^3 \rtimes S_4 & \text{Out}C_{10}^4 \simeq S_3 \times \mathbb{Z}_2 \\
\text{Out}C_3^4 \simeq \mathbb{Z}_2^3 \rtimes S_4 & \text{Out}C_{11}^4 \simeq S_3 \times \mathbb{Z}_2 \\
\text{Out}C_4^4 \simeq \mathbb{Z}_2^3 \rtimes D_8 & \text{Out}C_{12}^4 \simeq D_8 \\
\text{Out}C_5^4 \simeq S_4 \times \mathbb{Z}_2 & \text{Out}C_{13}^4 \simeq S_4 \\
\text{Out}C_6^4 \simeq GL_3(2) & \text{Out}C_{14}^4 \simeq \mathbb{Z}_2^3 \rtimes S_3 \\
\text{Out}C_7^4 \simeq S_4 & \text{Out}C_{15}^4 \simeq \mathbb{Z}_2^3 \rtimes S_3 \\
\text{Out}C_8^4 \simeq S_4 & \text{Out}C_{16}^4 \simeq \mathbb{Z}_2^3
\end{array}$$

Proof. 1. Let (a, b, c, d) be a basis of the loop C_1^4 with characteristic vector $\lambda_1^4 = (1110110100)$, hence we have $a^2 = b^2 = c^2 = d^2 = -1$ and $[a, b] = [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = -1$. We also have that (a, b, c) is a basis of the code loop of rank 3 with characteristic vector $\lambda_1^3 = (111111)$.

Let $\varphi \in \text{Stab}(\lambda_1^3)$. We extend φ to $\tilde{\varphi}$ defining:

$$\begin{array}{ll}
\tilde{\varphi}: & a \quad a^\varphi d^{\alpha_1} = u \\
& b \longmapsto b^\varphi d^{\alpha_2} = v \\
& c \quad c^\varphi d^{\alpha_3} = w \\
& d \quad d
\end{array}, \quad \alpha_i \in \{0, 1\}, i = 1, 2, 3.$$

The matrix which represents $\tilde{\varphi}$ is of the form

$$\tilde{\varphi} = \left(\begin{array}{ccc|c} & & & \alpha_1 \\ & \varphi & & \alpha_2 \\ & & & \alpha_3 \\ \hline 0 & 0 & 0 & 1 \end{array} \right).$$

Clearly $\tilde{\varphi} \in GL_4(2)$, so it remains to prove that $\tilde{\varphi} \in \text{Stab}(\lambda_1^4)$, that is, $(\lambda_1^4)^{\tilde{\varphi}} = \lambda_1^4$. For this, we should find for which $(\alpha_1, \alpha_2, \alpha_3)$ we have the following relations:

- (a) $u^2 = v^2 = w^2 = 1$
- (b) $[u, v] = [u, w] = [v, w] = -1$ and $[u, d] = [v, d] = [w, d] = 1$

Let's see:

- (a) Case $\alpha_i = 0$: $(a^\varphi)^2 = (b^\varphi)^2 = (c^\varphi)^2 = -1$, since $\varphi \in \text{Stab}(\lambda_1^3)$.

Case $\alpha_i = 1$: $(a^\varphi d)^2 = (a^\varphi)^2 d^2 [a^\varphi, d] = -1$, because $(a^\varphi)^2 = -1$ ($\varphi \in \text{Stab}(\lambda_1^3)$) and $d^2 = [a^\varphi, d] = 1$ ($[a, d] = [b, d] = [c, d] = 1$).

- (b) Clearly, we have $[u, d] = [v, d] = [w, d] = 1$. Now, we will see that $[u, v] = [u, w] = [v, w] = -1$. If $\alpha_i = 0$, then we have $[a^\varphi, b^\varphi] = -1$, because

$\varphi \in \text{Stab}(\lambda_1^3)$. If $\alpha_i = 1$, we have $[a^\varphi d, b^\varphi d] = [a^\varphi, b^\varphi d][d, b^\varphi d](a^\varphi, b^\varphi, d)$, but $[d, b^\varphi d] = (a^\varphi, b^\varphi, d) = 1$, then $[a^\varphi d, b^\varphi d] = [a^\varphi, b^\varphi][a^\varphi, d] = [a^\varphi, b^\varphi] = -1$. Hence, $[u, v] = -1$. Analogously, we prove the other two cases.

Therefore, for all $\varphi \in \text{Stab}(\lambda_1^3)$ and for any values of $(\alpha_1, \alpha_2, \alpha_3)$, we have $\tilde{\varphi} \in \text{Stab}(\lambda_1^4)$. By the Proposition 6 (groups of automorphisms of code loops of rank 3), we have $|\text{Stab}(\lambda_1^3)| = 168$. Hence $|\text{Stab}(\lambda_1^4)| = 168 \cdot 8 = 1344$.

We consider the following stabilizers of λ_1^4 :

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \sigma_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_5 &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_6 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \rho_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_3 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We have $\sigma_i^2 = 2$, $i = 1, \dots, 6$, $\rho_i^2 = 1$, $\rho_i \rho_j = \rho_j \rho_i$, $i, j = 1, 2, 3$. Let $H = GR\{\sigma_i, i = 1, \dots, 6\}$ and $N = GR\{\rho_i, i = 1, 2, 3 | \rho_i^2 = 1, \rho_i \rho_j = \rho_j \rho_i\}$, be subgroups of $\text{Stab}(\lambda_1^4)$. We know that $H \simeq GL_3(2)$, $N \simeq \mathbb{Z}_2^3$, and we still have $\text{Stab}(\lambda_1^4) = HN$ and $H \cap N = \{1\}$. Besides, with direct calculations we prove that N is normal in $\text{Stab}(\lambda_1^4)$. Therefore, $\text{Out}(C_1^4) \simeq \mathbb{Z}_2^3 \rtimes GL_3(2)$.

2. In this case $\lambda_2^4 = (0000000000)$, that is, $a^2 = b^2 = c^2 = d^2 = 1$ and $[a, b] = [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = 1$. The basis (a, b, c) give us the code loop of rank 3, with $\lambda_2^3 = (000000)$.

Let $\varphi \in \text{Stab}(\lambda_2^3)$. We extend φ to $\tilde{\varphi} \in GL_4(2)$, as in the previous case. We must prove that $\tilde{\varphi} \in \text{Stab}(\lambda_2^4)$, that is, $(\lambda_2^4)^{\tilde{\varphi}} = \lambda_2^4$. In fact, since $\varphi \in \text{Stab}(\lambda_2^3)$, then

(a) $(a^\varphi d^{\alpha_1})^2 = (a^\varphi)^2 (d^{\alpha_1})^2 [a^\varphi, d^{\alpha_1}] = 1$. Analogously, $(b^\varphi d^{\alpha_2})^2 = (c^\varphi d^{\alpha_3})^2 = 1$.

(b) Clearly, $[a^\varphi d^{\alpha_1}, d] = [b^\varphi d^{\alpha_2}, d] = [c^\varphi d^{\alpha_3}, d] = 1$.

Now, we have that $[a^\varphi d^{\alpha_1}, b^\varphi d^{\alpha_2}] = [a^\varphi, b^\varphi d^{\alpha_2}][d^{\alpha_1}, b^\varphi d^{\alpha_2}](a^\varphi, d^{\alpha_1}, b^\varphi d^{\alpha_2})$. Since $[d^{\alpha_1}, b^\varphi d^{\alpha_2}] = (a^\varphi, d^{\alpha_1}, b^\varphi d^{\alpha_2}) = 1$, and $[a^\varphi, b^\varphi] = [a^\varphi, d^{\alpha_2}] = (a^\varphi, b^\varphi, d^{\alpha_2}) = 1$, we obtain $[a^\varphi d^{\alpha_1}, b^\varphi d^{\alpha_2}] = 1$. Similarly, $[a^\varphi d^{\alpha_1}, c^\varphi d^{\alpha_3}] = [b^\varphi d^{\alpha_2}, c^\varphi d^{\alpha_3}] = 1$.

We proved that, $\tilde{\varphi} \in \text{Stab}(\lambda_2^4)$, for all $\varphi \in \text{Stab}(\lambda_2^3)$ and for any values of $(\alpha_1, \alpha_2, \alpha_3)$.

By the Proposition 6, we have $|\text{Stab}(\lambda_2^3)| = 24$. Hence, $|\text{Stab}(\lambda_2^4)| = 24 \cdot 8 = 192$.

To prove that $\text{Out}C_2^4 \simeq \mathbb{Z}_2^3 \rtimes S_4$, we just need to consider the subgroups H and N of $\text{Stab}(\lambda_2^4)$ given respectively by:

$$H = GR\{\sigma_i, i = 1, 2, 3 | \sigma_i^2 = 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_1 \sigma_3 = \sigma_3 \sigma_1\} \simeq S_4$$

and

$$N = GR\{\rho_i, i = 1, 2, 3 | \rho_i^2 = 1, \rho_i \rho_j = \rho_j \rho_i\} \simeq \mathbb{Z}_2^3.$$

In this case, we consider the generators these groups given as follows:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \rho_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_3 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We have that $\text{Stab}(\lambda_2^4) = HN$, $H \cap N = \{1\}$ and N is a normal subgroup of $\text{Stab}(\lambda_2^4)$.

3. In this case $\lambda_3^4 = (0000110100)$ and $\lambda_3^3 = (000111)$.

We have $(a^{\tilde{\varphi}})^2 = (a^{\varphi} d^{\alpha_1})^2 = (a^{\varphi})^2 (d^{\alpha_1})^2 [a^{\varphi}, d] = 1$. Analogously, we have $(b^{\tilde{\varphi}})^2 = (c^{\tilde{\varphi}})^2 = 1$. Then, for any triple $(\alpha_1, \alpha_2, \alpha_3)$, we have $\tilde{\varphi} \in \text{Stab}(\lambda_3^4)$. Besides, we know that $|\text{Stab}(\lambda_3^3)| = 24$, hence $|\text{Stab}(\lambda_3^4)| = 24 \cdot 8 = 192$.

Analogously to previous case, we prove that $\text{Out}C_2^4 \simeq \mathbb{Z}_2^3 \rtimes S_4$. In this case, we consider the generators of the groups $H \simeq S_4$ and $N \simeq \mathbb{Z}_2^3$, given by:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \rho_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rho_3 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

4. Let $\lambda_4^4 = (0010100000)$ and $\lambda_4^3 = (001100)$. We have that $|\text{Stab}(\lambda_4^3)| = 8$ because $\lambda_4^3 \in O_4^3$. In fact, we assume $a^{\varphi} = x, b^{\varphi} = y, c^{\varphi} = z$. We have 8 possibilities for (x, y, z) , given by (a, b, c) , (b, a, c) , (a, b, ac) , (b, a, ac) , (a, b, bc) , (b, a, bc) , (a, b, abc) , (b, a, abc) . In any case $x^2 = y^2 = 1, z^2 = -1, [x, y] = -1, [x, y] = [y, z] = 1$. Now,

we have $(a^{\tilde{\varphi}})^2 = (a^{\varphi}d^{\alpha_1})^2 = (a^{\varphi})^2(d^{\alpha_1})^2[a^{\varphi}, d] = 1$. Analogously, we have $(b^{\tilde{\varphi}})^2 = 1$ and $(c^{\tilde{\varphi}})^2 = -1$. Then, for any triple $(\alpha_1, \alpha_2, \alpha_3)$, we have $\tilde{\varphi} \in \text{Stab}(\lambda_4^4)$. Hence $|\text{Stab}(\lambda_4^4)| = 8.8 = 64$.

We consider the following stabilizers of λ_4^4 :

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With direct calculations we obtain: $\sigma_1^4 = 1$, $\sigma_2^2 = 1$, $\sigma_2\sigma_1\sigma_2 = \sigma_1^3$, $\rho_1^2 = \rho_2^2 = \rho_3^2 = 1$, $\rho_i\rho_j = \rho_j\rho_i$, $i, j = 1, 2, 3$.

Let $K_1 = GR\{\sigma_1, \sigma_2 \mid \sigma_1^4 = 1, \sigma_2^2 = 1, \sigma_2\sigma_1\sigma_2 = \sigma_1^3\}$ and $H_1 = GR\{\rho_1, \rho_2, \rho_3 \mid \rho_1^2 = \rho_2^2 = \rho_3^2 = 1, \rho_i\rho_j = \rho_j\rho_i, i, j = 1, 2, 3\}$. The elements these two groups are also stabilizers of λ_4^4 . Indeed, the product of any element of K_1 by H_1 is a stabilizer of λ_4^4 . Hence we have $H_1 \cap K_1 = \{1\}$ and $\text{Stab}(\lambda_4^4) = K_1 H_1$.

We know that $K_1 \simeq D_8$ and $H_1 \simeq \mathbb{Z}_2^3$. We still have that K_1 isn't a normal subgroup of $\text{Stab}(\lambda_4^4)$, because, for example, for $\rho_1 \in H_1$ and $\sigma_1 \in K_1$ we don't have $\rho_1\sigma_1\rho_1 \in K_1$, that is, $\rho_1 K_1 \rho_1 \not\subseteq K_1$. For the other side, $H_1 \triangleleft \text{Stab}(\lambda_4^4)$. To prove this, we just do the calculations directly and we use the following relations:

$$\sigma_2\rho_1 = \rho_1\sigma_2, \sigma_2\rho_2 = \rho_2\sigma_2, \sigma_1\rho_3 = \rho_3\sigma_1, \sigma_2\rho_3 = \rho_3\sigma_2, \rho_1\sigma_1 = \sigma_1\rho_2, \sigma_1\rho_1\rho_3 = \rho_2\sigma_1, \\ \rho_1\rho_2\sigma_1 = \sigma_1\rho_2\rho_1\rho_3, \sigma_1\rho_1 = \rho_2\rho_3\sigma_1, \sigma_1\rho_2\rho_3 = \rho_1\rho_3\sigma_1, \sigma_1\rho_1\rho_2 = \rho_2\rho_3\rho_1\sigma_1.$$

Therefore, $\text{Stab}(\lambda_4^4) \simeq \mathbb{Z}_2^3 \rtimes D_8$.

5. Let $\lambda_5^4 = (0000010100)$ and $\lambda_5^3 = (000011)$. In this case, $|\text{Stab}(\lambda_5^3)| = 6$ because $\lambda_5^3 \in O_5^3$. The possibilities for $(x = a^{\varphi}, y = b^{\varphi}, z = c^{\varphi})$ are (a, b, c) , (a, ab, c) , (b, a, c) , (b, ab, c) , (ab, a, c) , (ab, b, c) . Clearly, $|\text{Stab}(\lambda_5^4)| = 6.8 = 48$.

We consider

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We have that ρ and σ_i , $i = 1, 2, 3$, are stabilizers of λ_5^4 of order 2. We still have $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ and $\rho\sigma = \sigma\rho$, for each $\sigma \in S_4$. The group generated by σ_i , $i = 1, 2, 3$ is isomorphic to S_4 , $GR\{\rho\} \simeq \mathbb{Z}_2$. Therefore, $\text{Out}C_5^4 \simeq S_4 \times \mathbb{Z}_2$.

6. Let $\lambda_6^4 = (1111110100)$ and $\lambda_1^3 = (111111)$. We have $|\text{Stab}(\lambda_1^3)| = 168$. In this case, only for $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ we will have $\tilde{\varphi} \in \text{Stab}(\lambda_6^4)$. In fact, $(a^\varphi d)^2 = (b^\varphi d)^2 = (c^\varphi d)^2 = 1$, since $(a^\varphi)^2 = (b^\varphi)^2 = (c^\varphi)^2 = -1$ and $d^2 = -1$. Therefore, $|\text{Stab}(\lambda_6^4)| = 168$. We note that $a^\varphi, b^\varphi, c^\varphi \in \{a, b, c, ab, ac, bc, abc\}$.

Let the following stabilizers of λ_6^4 : $\sigma_1, \sigma_2, \dots, \sigma_6$, represented in matrix form by:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \sigma_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We have $a^{\sigma_1} = ab$, $b^{\sigma_1} = b$, $c^{\sigma_1} = c$ and $d^{\sigma_1} = d$. Hence $\sigma_1 \in \text{Stab}(\lambda_6^4)$. Analogously, we see that $\sigma_2, \dots, \sigma_6 \in \text{Stab}(\lambda_6^4)$. Any product of the σ_i 's, $i = 1, \dots, 6$ is a stabilizer of λ_6^4 . Hence $GR\{\sigma_1, \dots, \sigma_6\} = \text{Stab}(\lambda_6^4)$, but $GR\{\sigma_1, \dots, \sigma_6\} \simeq GL_3(2)$, therefore, $\text{Stab}(\lambda_6^4) \simeq GL_3(2)$.

7. In this case, we have $\lambda_7^4 = (0001000000)$, $\lambda_2^3 = (000000)$ and $|\text{Stab}(\lambda_2^3)| = 24$. Analogously to the previous case, we only have $\tilde{\varphi} \in \text{Stab}(\lambda_7^4)$ for $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. Therefore, $|\text{Stab}(\lambda_7^4)| = 24$.

$$\text{Let } \sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \text{ We have:}$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2; \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2 \text{ and } \sigma_1\sigma_3 = \sigma_3\sigma_1.$$

As $\sigma_1, \sigma_2, \sigma_3 \in \text{Stab}(\lambda_7^4)$, then $\text{Stab}(\lambda_7^4) = GR\{\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2, \sigma_1\sigma_3 = \sigma_3\sigma_1\}$, but the group generated by $\sigma_1, \sigma_2, \sigma_3$ and which satisfies the relations above is isomorphic to the permutation group of 4 elements, S_4 . Therefore, $\text{Stab}(\lambda_7^4) \simeq S_4$.

8. Let $\lambda_8^4 = (0000001000)$ and $\lambda_2^3 = (000000)$. We suppose that $\sigma \in \text{Stab}(\lambda_8^4)$ and $a^\sigma = u, b^\sigma = v, c^\sigma = w, d^\sigma = d$. Then

$$u^2 = v^2 = w^2 = 1; \quad d^2 = 1 \quad (21)$$

$$[u, v] = [u, w] = 1 \quad (22)$$

$$[v, w] = [v, d] = [w, d] = 1 \quad (23)$$

$$[u, d] = -1 \quad (24)$$

By (21) and (24) we have $u \in \{a, ab, ac, abcd\}$ and by (21) and (23) we have $v, w \in \{b, c, bc, bd, cd, bcd\}$. Hence, by (22) we obtain all the possible possibilities for (u, v, w) . For example, for $u = a$ we have $(u, v, w) = (a, b, c), (a, b, bcd), (a, c, b), (a, c, bcd), (a, bcd, b)$ or (a, bcd, c) . For any u we obtain 6 different basis. Therefore, in total we have 24 different basis and thus, $|\text{Stab}(\lambda_8^4)| = 24$.

To prove that $\text{Out}(C_8^4) \simeq S_4$, we just need to consider the following stabilizers:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

9. Let $\lambda_9^4 = (0100001000)$ and $\sigma \in \text{Stab}(\lambda_9^4)$ defined as above. In this case:

$$u^2 = w^2 = d^2 = 1; \quad v^2 = -1 \quad (25)$$

$$[u, v] = [u, w] = 1 \quad (26)$$

$$[v, w] = [v, d] = [w, d] = 1 \quad (27)$$

$$[u, d] = -1 \quad (28)$$

By (25) and (28), we have $u \in \{a, ac, abc, abd\}$ and by (25) and (27) we have $v \in \{b, bc, bd, bcd\}$ and $w \in \{c, cd\}$. Now, by (26) we have that (u, v, w) are given by $(a, b, c), (a, bcd, c), (ac, bc, c), (ac, bd, c), (abc, bd, cd), (abc, bcd, cd), (abd, b, cd), (abd, bc, cd)$. Therefore, $|\text{Stab}(\lambda_9^4)| = 8$.

Let $\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\rho = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. With direct calculations we obtain

that the orders of σ and ρ are 2 and 4, respectively, and the relation $\sigma\rho\sigma^{-1} = \rho^{-1}$ is valid, or better, $\rho\sigma = \sigma\rho^3$. We have that σ and ρ stabilizes λ_9^4 .

Hence $GR\{\sigma, \rho \mid \rho^4 = 1, \sigma^2 = 1, \sigma\rho\sigma^{-1} = \rho^{-1}\} = \text{Stab}(\lambda_9^4)$. Therefore, $\text{Stab}(\lambda_9^4) \simeq D_8$.

10. Let $\lambda_{10}^4 = (0001111000)$ and $\lambda_{5''}^3 = (000110)$. We have $|\text{Stab}(\lambda_{5''}^3)| = 6$. In fact,

if $\varphi \in \text{Stab}(\lambda_{5''}^3)$, then the possibilities for $\lambda_{5''}^3$ are $(a^\varphi, b^\varphi, c^\varphi) = (a, bc, c)$, (a, c, bc) , (a, b, bc) , (a, bc, b) , (a, c, b) or (a, b, c) .

Let $\tilde{\varphi}$ as before and we suppose that $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. In this case, we have 6 different stabilizers of λ_{10}^4 of the form $(a^\varphi, b^\varphi, c^\varphi, d)$, since $a^\varphi = a$, $b^\varphi = b, c$ or bc and $c^\varphi = b, c$ or bc . Now, assuming $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$, we have that $\tilde{\varphi}$ is $(a^{\tilde{\varphi}}, b^{\tilde{\varphi}}, c^{\tilde{\varphi}}, d^{\tilde{\varphi}}) = (a^\varphi d, b^\varphi, c^\varphi, d)$ and hence, $(a^\varphi d)^2 = (b^\varphi)^2 = (c^\varphi)^2 = 1$, $[a^\varphi d, b^\varphi] = [a^\varphi d, c^\varphi] = [a^\varphi d, d] = -1$ and $[b^\varphi, d] = [c^\varphi, d] = [b^\varphi, c^\varphi] = 1$. Hence, we have more 6 stabilizers of λ_{10}^4 .

For any other triple $(\alpha_1, \alpha_2, \alpha_3)$ we will not have more stabilizers of λ_{10}^4 , because if, for example, $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$ we will have $b^{\tilde{\varphi}} = b^\varphi d$, but $(b^\varphi d)^2 = -1$ since $d^2 = -1$ and $(b^\varphi)^2 = [b^\varphi, d] = 1$. The analysis is analogous for the other possibilities of $(\alpha_1, \alpha_2, \alpha_3)$. Therefore, $|\text{Stab}(\lambda_{10}^4)| = 12$.

We consider the following stabilizers of λ_{10}^4 :

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We have $\sigma_1^2 = \sigma_2^2 = \rho^2 = 1$ and $\sigma\rho = \rho\sigma$, for each $\sigma \in S_3 \cdot \mathbb{Z}_2$.

Therefore, $\text{Out}C_{10}^4 \simeq S_3 \times \mathbb{Z}_2$.

11. Let $\lambda_{11}^4 = (0001001000)$. We have $u^2 = v^2 = w^2 = 1, d^2 = -1$, $[u, v] = [u, w] = 1$, $[v, w] = [v, d] = [w, d] = 1$ and $[u, d] = -1$. Thus, we obtain $u \in \{a, ab, ac, ad, abd, acd\}$ and $v, w \in \{b, c, bc\}$. Hence, $|\text{Stab}(\lambda_{11}^4)| = 12$.

We consider the following stabilizers of λ_{11}^4 :

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \theta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ e } \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With simple calculations we obtain $\sigma^2 = \theta^2 = \rho^2 = 1$, $\sigma\theta\sigma = \theta\sigma\theta$, $\sigma\theta \neq \theta\sigma$, $\sigma\rho = \rho\sigma$, $\theta\rho = \rho\theta$, $\sigma\theta\rho = \rho\sigma\theta$, $\sigma\theta\sigma\rho = \rho\sigma\theta\sigma$ e $\theta\sigma\rho = \rho\theta\sigma$. All these elements are stabilizers of λ_{11}^4 . Let $H = GR\{\sigma, \theta \mid \sigma^2 = \theta^2, \sigma\theta\sigma = \theta\sigma\theta\}$ and $K = GR\{\rho \mid \rho^2 = 1\}$. Thus, from the before calculations, any stabilizer of λ_{11}^4 is written uniquely as the product of an element of H by an element of K and $xy = yx$, for all $x \in H, y \in K$. Besides, we know that $H \cong S_3$ and $K \cong \mathbb{Z}_2$. Therefore, we have $\text{Stab}(\lambda_{11}^4) \simeq S_3 \times \mathbb{Z}_2$.

12. Let $\lambda_{12}^4 = (0000001100)$. We have $u^2 = v^2 = w^2 = d^2 = 1$, $[u, v] = [u, w] = [v, d] =$

$[w, d] = 1$ and $[u, d] = [v, w] = -1$. Thus, we obtain $u \in \{a, ab, ac, abc\}$ and $v, w \in \{b, c\}$. Hence, $|\text{Stab}(\lambda_{12}^4)| = 8$.

In this case, we consider $\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\rho = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

stabilizers of $\text{Stab}(\lambda_{12}^4)$. Analogously the analysis of the case 9, we have $GR\{\sigma, \rho \mid \rho^4 = 1, \sigma^2 = 1, \sigma\rho\sigma^{-1} = \rho^{-1}\} = \text{Stab}(\lambda_{12}^4)$. Therefore, $\text{Stab}(\lambda_{12}^4) \simeq D_8$.

13. Let $\lambda_{13}^4 = (0110111100)$. In this case, $u^2 = d^2 = 1$, $v^2 = w^2 = -1$, $[u, v] = [u, w] = [u, d] = [v, w] = -1$, $[v, d] = [w, d] = 1$. Hence, we have $u \in \{a, ab, ac, abc\}$ and $v, w \in \{b, c, bc\}$. Thus, $|\text{Stab}(\lambda_{13}^4)| = 24$.

Here the required generators are given by:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Analogously to the previous cases, we prove that $\text{Out}(C_{13}^4) \simeq S_4$.

14. For $\lambda_{14}^4 = (0001001100)$ we have $u^2 = v^2 = w^2 = 1$, $d^2 = -1$, $[u, v] = [u, w] = 1$, $[v, d] = [w, d] = 1$ and $[u, d] = [v, w] = -1$. Hence, $u \in \{a, ab, ac, ad, abc, abd, acd, abcd\}$ and $v, w \in \{b, c, bcd\}$. Therefore, $|\text{Stab}(\lambda_{14}^4)| = 48$.

We consider the following generators for \mathbb{Z}_2^3 :

$$\rho_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For S_3 , we consider: $\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

With the required calculations we obtain $\text{Out}C_{14}^4 \simeq \mathbb{Z}_2^3 \rtimes S_3$.

15. For $\lambda_{15}^4 = (1001001100)$ we have $u^2 = -1$, $v^2 = w^2 = 1$, $d^2 = -1$, $[u, v] = [u, w] = [v, d] = [w, d] = 1$ and $[u, d] = [v, w] = -1$. Thus, $u \in \{a, ab, ac, ad, abc, abd, acd, abcd\}$ and $v, w \in \{b, c, bcd\}$. Therefore, $|\text{Stab}(\lambda_{15}^4)| = 48$.

We consider the following generators for \mathbb{Z}_2^3 :

$$\rho_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For S_3 , we consider: $\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

In this case, we also obtain $\text{Out}C_{15}^4 \simeq \mathbb{Z}_2^3 \rtimes S_3$.

16. Let $\lambda_{16}^4 = (0001111100)$, then $u^2 = v^2 = w^2 = 1$, $d^2 = -1$, $[u, v] = [u, w] = -1$, $[u, d] = [v, w] = -1$ and $[v, d] = [w, d] = 1$, and hence, $u \in \{a, abc, ad, abcd\}$ and $v, w \in \{b, c\}$. Therefore, $|\text{Stab}(\lambda_{16}^4)| = 8$.

Let the following stabilizers of λ_{16}^4 :

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \theta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \rho = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $\sigma^2 = \theta^2 = \rho^2 = 1$ and $\sigma\theta = \theta\sigma$, $\sigma\rho = \rho\sigma$ and $\sigma\rho = \rho\sigma$. Clearly, $\text{Stab}(\lambda_{16}^4) \simeq \mathbb{Z}_2^3$. \square

4 Representations of Code Loops

First of all, we defined code loops using double even codes fixed. Now, we fix a code loop L and we want to determine the double even codes V such that $L \simeq L(V)$.

A representation of a given code loop L is a double even code $V \subseteq \mathbf{F}_2^m$ such that $L \simeq L(V)$. The degree of a representation is the number m .

We notice that there are many different representations for a same code loop. We prove by the Theorems 5 and 6 below that, there are representations of nonassociatives code loops of rank 3 and 4 such that the degree of each representation is the smallest possible.

Definition 4. A representation V is called basic if the degree of V is minimal.

We identify the \mathbf{F}_2 -space \mathbf{F}_2^m as the set of all subsets of $I_m = \{1, \dots, m\}$ and we define a relation of equivalence \sim on I_m : $i \sim j$ if and only if $\{i, j\} \cap v = \{i, j\}$ or $\{i, j\} \cap v = \emptyset$, for all $v \in V$.

We notice that this definition is equivalent to: $i \sim j$ if and only if $\{i, j\} \cap v_k = \{i, j\}$ or $\{i, j\} \cap v_k = \emptyset$, $k = 1, \dots, s$ and $\{v_1, \dots, v_s\}$ is a basis of V .

We will consider only representations such that, for any equivalence classes X , we have $|X| < 8$. We call this representations by reduced representations.

Our main problem is to find all basic representations for a given code loop.

Definition 5. For a given representation V and all the equivalence classes X_1, \dots, X_r , the type of V is a vector $(|X_1|, \dots, |X_r|)$ such that $|X_1| \leq |X_2| \leq \dots \leq |X_r|$.

Definition 6. Let V_1 and V_2 be double even codes of \mathbf{F}_2^m . We say that V_1 and V_2 are isomorphic even codes if and only if there is a bijection $\varphi \in S_m$ such that $V_1^\varphi = V_2$.

Theorem 5. The code loops C_1^3, \dots, C_5^3 have the following basic representations V_1, \dots, V_5 , which are given by

$$V_1 = \langle (1234), (1256), (1357) \rangle,$$

$$V_2 = \langle (12345678), (12349, 10, 11, 12), (15679, 10, 11, 13) \rangle,$$

$$V_3 = \langle (12345678), (1234569, 10), (1234579, 11) \rangle,$$

$$V_4 = \langle (1234), (1256789, 10, 11, 12, 13, 14), (1356789, 10, 11, 15, 16, 17) \rangle,$$

$$V_5 = \langle (12345678910, 11, 12), (1-8, 13, 14, 15, 16), (1-5, 9, 10, 11, 13, 14, 15, 17) \rangle.$$

Proof. We consider the \mathbf{F}_2 -subspaces V_1, \dots, V_5 of $\mathbf{F}_2^7, \mathbf{F}_2^{13}, \mathbf{F}_2^{11}, \mathbf{F}_2^{17}, \mathbf{F}_2^{17}$, respectively, as above. First we will see that each space V_i is a representation of C_i^3 , that is, is a double even code of \mathbf{F}_2^m , for some m and that $C_i^3 \simeq L(V_i)$, $i = 1, \dots, 5$.

Since the elements of V_5 are $v_0 = 0$, $v_1 = (12345678910, 11, 12)$, $v_2 = (12345678, 13, 14, 15, 16)$, $v_3 = (12345, 9, 10, 11, 13, 14, 15, 17)$, $v_4 = v_1 + v_2 = (9-16)$, $v_5 = v_1 + v_3 = (678, 12, 13, 14, 15, 17)$, $v_6 = v_2 + v_3 = (6789, 10, 11, 16, 17)$ and $v_7 = v_1 + v_2 + v_3 = (12345, 12, 16, 17)$, we see, clearly, that all the vectors have weight with multiplicity 4 and the weight of the intersection of each pair of vectors is even. Thus, V_5 is a double even code. Analogously, we prove that V_i , $i = 1, \dots, 4$ is a double even code.

Now, the isomorphism $C_i^3 \simeq L(V_i)$ follows directly from Theorem 3 of Classification of Code Loops of rank 3 and from Proposition 5. We just need to calculate the characteristic vector associated to $L(V_i)$ and note that it belongs to the orbit O_i^3 corresponding to the code loop C_i^3 .

As example, we calculate the characteristic vector associate to $L(V_5) = \{1, -1\} \times V_5$. We have $v_i^2 = (-1)^{\frac{|v_i|}{4}} = -1$ and $[v_i, v_j] = (-1)^{\frac{|v_i \cap v_j|}{2}} = 1$, for $i, j = 1, 2, 3$. Hence $\lambda(L(V_5)) = (111000) \in O_5^3$. Therefore, $C_5^3 \simeq L(V_5)$.

We will demonstrate now that each V_i , $i = 1, \dots, 5$, up to isomorphism, is the unique basic representation of the code loop C_i^3 . We consider $X = (a, b, c)$ a set of generators for C_i^3 such that $\lambda = \lambda_X(C_i^3)$ is the corresponding characteristic vector, and we assume that V is a basic representation of C_i^3 , where v, w, u are the elements of the basis of V which corresponds to a, b, c . We use the notation $t = |v \cap w \cap u|$. We remember that $t \equiv 1 \pmod{2}$, because $(a, b, c) = -1$.

Case $i = 1$: In this case, we are assuming $gr V \leq 7$. The characteristic vector is $\lambda = (111111)$, then $|v| \equiv |w| \equiv |u| \equiv 4 \pmod{8}$. We suppose that $v = (1234)$, but $[a, b] = -1$, then $|v \cap w| \equiv 2 \pmod{4}$, and hence, $|v \cap w| = 2$. Analogously, we obtain $|v \cap u| = 2$. Let $w = (1256)$, hence we also obtain $|w \cap u| = 2$ and then, $t = 1$. Hence, $u = (1357)$. Therefore,

$V = V_1$.

Case $i = 2$: The characteristic vector is $\lambda = (000000)$, then $|v| \equiv |w| \equiv |u| \equiv 0(\text{mod } 8)$. Since $\text{gr } V \leq 13$, then $|v| = 8$. Analogously, $|w| = |u| = 8$. Let $v = (12345678)$. As $|v \cap w| \equiv 0(\text{mod } 4)$, hence $|v \cap w| = 4$. Analogously, $|v \cap u| = |w \cap u| = 4$.

Let $w = (12349, 10, 11, 12)$. We have two possibilities for t : $t = 1$ or $t = 3$. Case $t = 3$, we will have $\text{gr } V = 15$, a contradiction. Hence, $t = 1$ and $u = (15679, 10, 11, 13)$. Therefore, $V = V_2$.

Case $i = 3$: We consider $\lambda = (000111)$ and $\text{gr } V \leq 11$. Analogously to the previous case, we have $|v| = |w| = |u| = 8$. In this case, we have $|v \cap w| = |v \cap u| = |w \cap u| = 6$. We suppose $v = (12345678)$ and $w = (1234569, 10)$. We have three possibilities for t : $t = 1, 3$ or 5 . Case $t \leq 3$ we will have $|v| > 8$, a contradiction. Hence, $t = 5$ and we can assume $u = (1234579, 11)$. Therefore, $V = V_3$.

Case $i = 4$: In this case, $\lambda = (111110)$. Since $|v| \equiv |w| \equiv |u| \equiv 4(\text{mod } 8)$ and $|w \cap u| \equiv 0(\text{mod } 4)$ then $|w| = |u| = 12$ and $|w \cap u| = 8$. Let $v = (1234)$, so $|v \cap w| = 2$. If $|v \cap w| = 4$, then $[v, w] = (-1)^{\frac{|v \cap w|}{2}} = 1$, which does not occur since $[a, b] = -1$.

Let $w = (1256789, 10, 11, 12, 13, 14)$. Since $|v \cap w| = 2$, then $t = 1$. Hence $u = (1356789, 10, 11, 15, 16, 17)$. Therefore, $V = V_4$.

Now, we suppose $v = (123456789, 10, 11, 12)$. If $|v \cap w| \leq 6$ then $\text{gr } V \geq 18$. In fact, we have $|v \cap w| = 2, 6$ or 10 . If $|v \cap w| = 2$ we will have $|v + w| = 20$, a contradiction. If $|v \cap w| = 6$ we will have $|v + w| = 12$ and hence, $\text{gr } V \geq 18$, a contradiction. Therefore, $|v \cap w| = 10$. Analogously we obtain $|v \cap u| = 10$. Without loss of generality, we suppose $w = (1256789, 10, 11, 12, 13, 14)$. Since $t \equiv 1(\text{mod } 2)$, then the possibilities for t are $1, 3, 4$ or 7 . In any case we will have $|v| \geq 13$, which is a contradiction. Then, there is not this last possibility for v .

Case $i = 5$: The characteristic vector is $\lambda = (111000)$, so we have $|v| \equiv |w| \equiv |u| \equiv 4(\text{mod } 8)$. Assuming $v = (1234)$ and since $[a, b] = 1$, we will have $|v \cap w| \equiv 0(\text{mod } 4)$. Case $v \cap w \neq \emptyset$ we will have $|v \cap w| = 4$, and hence $v \subset w$, which is a contradiction, because this give us $t \equiv 0(\text{mod } 2)$, which does not occur. The case $v \cap w = \emptyset$ also does not occur, since we must have nonempty intersection between v, w and u .

Therefore $|v| \geq 12$. Analogously, we prove that $|w| \geq 12$ and $|u| \geq 12$.

The representation V is basic and $\text{gr } V \leq 17$, so we obtain $|v| = |w| = |u| = 12$. Without loss of generality, we suppose $v = (123456789, 10, 11, 12)$ and $|w \cap v| = 4$ or 8 . If $|v \cap w| = 4$, then $|v + w| = 16$ and hence, $\text{gr } V \geq |v \cap w| + |v + w| = 20$, which is a contradiction.

Therefore, $v \cap w = (12345678)$ and $w = (12345678, 13, 14, 15, 16)$. Analogously, we have $|v \cap u| = |w \cap u| = 8$.

If $t \leq 3$, then $|v| \geq |v \cap w \cap u| + |(v \cap w) \setminus (v \cap w \cap u)| + |(v \cap u) \setminus (v \cap w \cap u)| \geq 3 + 5 + 5 = 13$,

a contradiction. Therefore, $t \geq 5$.

If $t = 7$, then $gr V \geq 19$. To prove this, we consider $u = (i_1, \dots, i_{12})$, where $i_1, \dots, i_7 \in v \cap w \cap u$, but since $v \cap w = (12345678)$ then $i_1, \dots, i_7 \in v \cap w$. Suppose $u = (1234567, i_8, \dots, i_{12})$. Since $|u \cap v| = 8$ and $|u \cap w| = 8$, so $i_8 \in \{9, 10, 11, 12\}$ whereas $i_9 \in \{13, 14, 15, 16\}$. We choose $i_8 = 10$ and $i_9 = 13$. Hence $i_j \notin \{9, 11, 12, 14, 15, 16\}$, $j = 10, 11, 12$. Hence, a possibility for u it will be $u = (1234567, 10, 13, 17, 18, 19)$ so that $gr V \geq 19$, a contradiction with the fact V be basic representation. Thus, $t = 5$ and $u = (123459, 10, 11, 13, 14, 15, 17)$. Therefore, $V = V_5$.

□

Corollary 4. *Each basic representation of the code loops C_1^3, \dots, C_5^3 has the following types, respectively: $(1111111), (1111333), (1111115), (1111337), (1113335)$.*

According with the Theorem 4, we have exactly 16 code loops of rank 4, namely, $C_1^4, C_2^4, \dots, C_{16}^4$. For each C_i^4 , $i = 1, \dots, 16$, we have to find $V_i \subseteq \mathbf{F}_2^m$ double even code of minimal degree m such that $V_i \cong L(C_i^4)$.

In general, the set $X = \{a, b, c, d\}$ represents a set of generates of C_i^4 such that $\lambda_X(C_i^4) = \lambda(C_i^4)$ is its corresponding characteristic vector. We also suppose that $V_i = ger \{v_1, v_2, v_3, v_4\}$ is a basic representation of C_i^4 , where v_1, v_2, v_3, v_4 corresponds to a, b, c, d respectively.

For the next theorem we use the notation: $t_{ijk} = |v_i \cap v_j \cap v_k|, i, j, k = 1, \dots, 4$ and $t_{1234} = |v_1 \cap v_2 \cap v_3 \cap v_4|$.

Theorem 6. *Each code loop C_1^4, \dots, C_{16}^4 has the following set of generators to its basic representation V_1, \dots, V_{16} , respectively:*

$$\begin{aligned}
V_1 &= \langle (1234), (1256), (1357), (1-8) \rangle, \\
V_2 &= \langle (1-8), (1-4, 9-12), (15679, 10, 11, 13), (12589, 12, 13, 14) \rangle, \\
V_3 &= \langle (1-8), (1-6, 9, 10), (1-5, 7, 9, 11), (1, 6-12) \rangle, \\
V_4 &= \langle (1-8), (1-6, 9, 10), (12379, 11-17), (1478, 9-11, 18) \rangle, \\
V_5 &= \langle (1-8), (1234, 9-12), (159, 13-17), (12569, 10, 13, 18) \rangle, \\
V_6 &= \langle (1234), (1256), (1357), (8, 9, 10, 11) \rangle, \\
V_7 &= \langle (1-8), (1234, 9-12), (15679, 10, 11, 13), (14, 15, 16, 17) \rangle, \\
V_8 &= \langle (1-8), (1234, 9-12), (12359, 13, 14, 15), (1, 2, 10, 11, 13, 14, 16, 17) \rangle, \\
V_9 &= \langle (1-8), (1234, 9-16), (15679, 10, 11, 17), (5, 6, 9, 10, 12, 13, 18, 19) \rangle, \\
V_{10} &= \langle (1-8), (1, 2, 9-14), (139, 10, 11, 15-17), (4, 5, 18, 19) \rangle, \\
V_{11} &= \langle (1-8), (1-4, 9-12), (12359, 13-15), (6, 7, 16, 17) \rangle, \\
V_{12} &= \langle (1-8), (1-4, 9-12), (1235, 9-11, 13), (129, 10, 14-17) \rangle, \\
V_{13} &= \langle (1-8), (129, 10), (139, 11), (45, 12-17) \rangle, \\
V_{14} &= \langle (1-8), (1-4, 9-12), (15679, 13-15), (2, 3, 10, 11) \rangle,
\end{aligned}$$

$$V_{15} = \langle (1-12), (1-4, 13-16), (1235, 13-17), (1-6, 13-18) \rangle, \\ V_{16} = \langle (1-8), (1, 2, 9-14), (1, 3-7, 9-13, 15-19), (1-5, 8, 9, 14-17, 20) \rangle.$$

Proof. Analogously to the Theorem 5 (Representations of the code loops of rank 3), we demonstrate that each V_i , $i = 1, \dots, 16$, is a double even code. Now, to prove that $V_i \simeq L(C_i^4)$ we just need to find the characteristic vector associated to $L(C_i^4)$ and apply the Theorem 4 (Classification of code loop of rank 4). Therefore, V_i is a representation of C_i^4 , $i = 1, \dots, 16$.

Now, we are going to prove, up to isomorphism, that V_1 is the unique basic representation of C_1^4 . We consider (a, b, c, d) a set of generators of C_1^4 such that $\lambda = \lambda(C_1^4) = (1110110100)$. We suppose that $V = \text{ger}\{v_1, v_2, v_3, v_4\}$ is a basic representation of C_1^4 , where v_1, v_2, v_3, v_4 corresponds to a, b, c, d , respectively. Hence, $\text{gr } V \leq 8$.

In this case, we have

$$\begin{aligned} |v_1| &\equiv |v_2| \equiv |v_3| \equiv 4(\text{mod } 8) \\ |v_4| &\equiv 0(\text{mod } 8) \\ |v_1 \cap v_2| &\equiv |v_1 \cap v_3| \equiv |v_2 \cap v_3| \equiv 2(\text{mod } 4) \\ |v_1 \cap v_4| &\equiv |v_2 \cap v_4| \equiv |v_3 \cap v_4| \equiv 0(\text{mod } 4) \end{aligned}$$

Suppose $v_1 = (1234)$, then $|v_1 \cap v_2| = |v_1 \cap v_3| = 2$ and hence, $t_{123} = 1$. Then we can assume $v_2 = (1256)$ and $v_3 = (1357)$.

We will analyse two possible cases for values of t_{1234} : 0 and 1.

We will write whenever necessary t_{ij4} , for $|v_i \cap v_j \cap v_4|$ with $i, j = 1, 2, 3$, $i \neq j$ and t_{1234} for $|v_1 \cap v_2 \cap v_3 \cap v_4|$.

Case $t_{1234} = 0$, we have $t_{ij4} = 0$ and thus, $|v_i \cap v_4| = 0$. Therefore, $|v_4| \geq 8$ and then, we don't have basic reduced representation in this case. Case $t_{1234} = 1$, we have $t_{ij4} = 2$ and thus, $|v_i \cap v_4| = 4$, that is, $v_i \subset v_4$, $i = 1, 2, 3$. Next, $v_4 = (12345678)$. Therefore, $V = V_1$.

If $v_1 = (123456789, 10, 11, 12)$ we have $\text{gr } V > 12$, which contradicts the hypothesis of V to be basic.

Now, we will prove that V_7 , up to isomorphism, is the unique basic representation of C_7^4 . Here the characteristic vector is given by $\lambda = (0001000000)$. We suppose that $V = \text{ger}\{v_1, v_2, v_3, v_4\}$ is a basic representation of C_7^4 . Then:

$$\begin{aligned} |v_1| &\equiv |v_2| \equiv |v_3| \equiv 0(\text{mod } 8) \\ |v_4| &\equiv 4(\text{mod } 8) \\ |v_1 \cap v_2| &\equiv |v_1 \cap v_3| \equiv |v_2 \cap v_3| \equiv 0(\text{mod } 4) \\ |v_1 \cap v_4| &\equiv |v_2 \cap v_4| \equiv |v_3 \cap v_4| \equiv 0(\text{mod } 4) \end{aligned}$$

Let $v_1 = (12345678)$, so $|v_1 \cap v_2| = |v_1 \cap v_3| = 4$ and hence, $t_{123} = 1$ or 3. Suppose $v_2 = (12349, 10, 11, 12)$, so $|v_2 \cap v_3| = 4$. Case $t_{123} = 1$, we consider $v_3 = (1567, 9, 10, 11, 13)$.

If $t_{1234} = 0$, then $t_{ij4} = 0$ or 2 . Considering that $|v_i \cap v_j| \equiv 0 \pmod{4}$, $i, j = 1, 2, 3$, $i \neq j$, then we have only two subcases for analyze:

- $t_{ij4} = 0$: In this subcase, $|v_i \cap v_4| = 0$, for $i = 1, 2, 3$ and hence, we can assume $v_4 = (14, 15, 16, 17)$. Thus, for this case, $V = V_7$.
- $t_{ij4} = 2$: Here, $|v_i \cap v_4| = 4$, for $i = 1, 2, 3$ and hence, $v_4 = (2, 3, 5, 6, 9, 10, 14 - 19)$, which contradicts the minimal degree of V .

If $t_{1234} = 1$, then $t_{ij4} = 2$ or 4 . Analogously, we have two subcases for analyze:

- $t_{ij4} = 2$: In this subcase, $|v_i \cap v_4| = 4$, for $i = 1, 2, 3$, which give us $gr V > 17$, a contradiction.
- $t_{ij4} = 4$: In this subcase, $|v_i \cap v_4| = 8$, for $i = 1, 2, 3$, which also contradicts the minimal degree of V .

Now, analyzing the case $t_{123} = 3$, we suppose that $v_3 = (12359, 13, 14, 15)$. If $t_{1234} = 0$, then $|v_i \cap v_4| = 0$ and hence, we will have $v_4 = (16, 17, 18, 19)$, which contradicts the degree of V to be minimal. Analogously, for the cases $t_{1234} = 1, 2$ and 4 , we will have $gr V > 17$.

Now, let the code loop C_{10}^4 with $\lambda = (0001111000)$ and $V = \text{ger}\{v_1, v_2, v_3, v_4\}$ its basic representation. Let $v_1 = (12345678)$, then $|v_1 \cap v_2| = 2$ or 6 and $|v_1 \cap v_3| = 2$ or 6 . Case $|v_1 \cap v_2| = 2$, we can assume $v_2 = (1, 2, 9 - 14)$. Hence, we have $t_{123} = 1$ and $|v_2 \cap v_3| = 4$.

- For $|v_1 \cap v_3| = 2$, consider $v_3 = (1, 3, 9, 10, 11, 15, 16, 17)$. If $t_{1234} = 0$, then $t_{124} = t_{134} = 0$ and $t_{234} = 0$ or 2 . Thus $|v_1 \cap v_4| = 2$. If $t_{234} = 0$: $|v_2 \cap v_4| = 0$ and $|v_3 \cap v_4| = 0$. Thus, we have $v_4 = (4, 5, 18, 19)$ and, therefore, $V = V_{10}$. In the case $t_{234} = 2$ we will find $v_4 = (4, 5, 9, 10, 12, 13, 15, 16, 18 - 21)$, contradicting the minimality of the degree of V . The analyze of $t_{1234} = 1$ is analogous.
- For $|v_1 \cap v_3| = 6$, we can consider $v_3 = (1, 3 - 7, 9 - 11, 15 - 21)$. In this case $gr V > 19$ for any analyze.

We don't have basic representation in case $|v_1 \cap v_2| = 6$.

Analogously, in the other cases, we prove that each V_i is the unique basic representation, up to isomorphism.

□

Corollary 5. *Each basic representation of the code loops C_1^4, \dots, C_{16}^4 has the following degree and type, respectively:*

i	$\deg V_i$	$\text{type of } V_i$	i	$\deg V_i$	$\text{type of } V_i$
1	8	(11111111)	2	14	(11111111222)
3	12	(111111114)	4	18	(11111111226)
5	18	(111111112224)	6	11	(11111114)
7	17	(11113334)	8	17	(11111122223)
9	19	(11111222233)	10	19	(111223333)
11	17	(111122333)	12	17	(1111112234)
13	17	(111111236)	14	15	(111112233)
15	18	(111111336)	16	20	(11111122334)

Note that in the case of code loops of rank 3 and 4 the type of code loop define this loop up to isomorphism. May be it is true in general case.

Conjecture 2. *Let V_1 and V_2 be representations of a code loop L . If this representations have the same degree and type, then V_1 and V_2 are isomorphic.*

References

- [1] R. M. PIRES, Loops de código: automorfismos e representações. 2011. 93f. Tese (Doutorado em Matemática)-Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2011.
- [2] R.L. Griess Jr., *Code loops*, J.Algebra 100 (1986),224–234.
- [3] R.H. Bruck, *A survey of binary systems*, Springer-Verlag (1958).
- [4] O. Chein and E.G. Goodaire, *Moufang Loops with a Unique Nonidentity Commutator (Associator, Square)*, J. Algebra 130 (1990), 369-384.
- [5] H.O. Pflugfelder, *Quasigroups and Loops: An Introduction*, Berkin:Heldermann (1990).
- [6] O. Chein, H.O. Pflugfelder, and J.D.H. Smith, *Quasigroups and Loops: Theory and Applications*, Berkin:Heldermann (1990).
- [7] E.G. Goodaire, E. Jaspers and C. Polcino, *Alternative Loop Rings*, North Holland Math, Studies N.184, Elsevier, Amsterdam (1996).